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Time-dependent backgrounds of 2D string theory: non-perturbative effects

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Abstract: We study the non-perturbative corrections (NPC) to the partition function of a compactified 2D string theory in a time-dependent background generated by a tachyon source. The sine-Liouville deformation of the theory is a particular case of such a background. We calculate the leading as well as the subleading NPC using the dual description of the string theory as matrix quantum mechanics. As in the minimal string theories, the NPC are classified by the double points of a complex curve. We calculate them by two different methods: by solving Toda equation and by evaluating the quasiclassical fermion wave functions. We show that the result can be expressed in terms of correlation functions of the bosonic field associated with the tachyon source and identify the leading and the subleading corrections as the contributions from the one-point (disk) and two-point (annulus) correlation functions.

Keywords: Bosonic Strings, D-branes, Matrix Models.

Contents

1. Introduction

The $c \leq 1$ string theories, which can also be viewed as solvable models of 2D quantum gravity, have been solved perturbatively through their description as large-N matrix models (see the reviews [1, 2, 3]). Originally the matrix models were considered just as 'engins' that generate planar graphs, but very soon it became clear that the matrix models are in principle able to give a qualitative picture of the non-perturbative effects[1] in these low-dimensional string theories [5].

[1]The first systematic study of the non-perturbative effects in the simplest one-matrix model has been done by F. David [4].

Recent works [6]–[21] suggested to think of the matrix models as dual open string theories in presence of D-branes and led to an interpretation of the non-perturbative phenomena in terms of open string world sheets with appropriate boundary conditions. This interpretation was prepared, from the string theory side, by the remarkable works on Liouville CFT with boundary, which has been solved using conformal bootstrap methods [22]–[27]. All comparisons between the matrix and CFT approaches showed agreement and led further to a new proposal for the matrix description of string theories with world sheet supersymmetry, type 0A and 0B theories [28, 29, 30].

In particular, the non-perturbative corrections (NPC) to the string partition function are believed to be produced by D-instantons [31]. On the string theory side, they were shown to be described in terms of a boundary CFT with Dirichlet boundary conditions on the Liouville field, known also as ZZ branes [24]. Namely, the leading corrections are given by the exponents of the disk partition functions with ZZ boundary conditions [32, 10, 11].

For $c < 1$ minimal string theories, the leading non-perturbative effects were given a nice geometric interpretation in terms of a complex curve describing the closed string theory background [12, 14]. Each point on this Riemann surface is associated with a FZZT brane, which implies Neumann boundary conditions for the Liouville field [22]. The ZZ branes are associated with the double points of the Riemann surface, which can be also thought of as vanishing A-cycles. The disk partition functions with ZZ boundary conditions are given by line integrals along the corresponding B-cycles.

The generalization of these results to $c = 1$ is not obvious because of the singular character of the limit $c \to 1$ in the space of minimal string theories. In particular, the double points of the complex curve degenerate just to two singularities where all ZZ branes are situated. Therefore, in order to study the non-perturbative effects in $c = 1$ string theories, it is helpful to perturb the theory with a time-dependent tachyon potential, to the effect that the degeneracy of the singular points at $c = 1$ is lifted.

The integrable tachyon perturbations are those with equidistant spectrum and it is natural to consider them in the context of a string theory with finite temperature. The world sheet description of such a theory is given by the euclidean $c = 1$ string theory, defined by the world sheet action

$$S_{c=1} = \frac{1}{4\pi} \int d^2\sigma \left[(\partial\chi)^2 + (\partial\phi)^2 + 2\hat{R}\phi + \mu\phi\, e^{2\phi} + \text{ghosts} \right], \tag{1.1}$$

in which the matter field χ, or the local euclidean time coordinate of the string, is compactified:

$$\chi + 2\pi R \equiv \chi\,. \tag{1.2}$$

We are interested in the possible deformations of this theory by on-mass-shell tachyon operators[2]

$$\mathcal{T}_p \sim \int d^2\sigma\, e^{ip\chi} e^{(2-|p|)\phi} \tag{1.3}$$

[2]In the compactified theory, in addition to this discrete spectrum of tachyon modes, there is a discrete set of the winding modes, or Kosterlitz-Thouless vortices. In the following we will consider only perturbations by tachyons; the results for perturbations by winding modes can be obtained by T-duality.

with the allowed by the compactification (1.2) values of the momentum

$$p_k = \frac{k}{R}, \qquad k = \pm 1, \pm 2, \dots . \tag{1.4}$$

A general such deformation is achieved by adding to the action (1.1) a term

$$\delta S = \sum_{k\neq 0} t_k \mathcal{T}_{k/R} . \tag{1.5}$$

We will be mainly interested in sine-Liouville deformation, which represents a perturbation by the two lowest vertex operators $n = \pm 1$. It contains, besides the Liouville term $\mu e^{2\phi}$, a sine-Liouville interaction

$$\delta S_{_{SL}} = \lambda \int d^2\sigma \, \cos\left(\frac{\chi}{R}\right) e^{(2-\frac{1}{R})\phi} , \tag{1.6}$$

where we denoted $\lambda = t_1 = t_{-1}$.

The string theory deformed by the term (1.5) has been studied using the 'holographic' description provided by the Matrix Quantum Mechanics (MQM). The first exact result, the expression for the partition function for sine-Liouville deformation, was suggested by G. Moore in [33]. Later this result was proved and generalised in [34, 35, 36] using the fact that the perturbations (1.5) behave as Toda integrable flows. The classical background for given deformation is described by a complex curve whose exact form depends on the coupling constants in (1.5). This curve is not the one describing the FZZT brane, but is obtained from the latter by a projection that identifies an infinite number of sheets. The existence of two distinct complex curves is a peculiarity of the $c = 1$ string theory and is related with the logarithmic singularity of the resolvent at infinity. In the $c < 1$ string theories there is only one such curve.

The leading non-perturbative effects in presence of sine-Liouville deformation were studied in [10, 11] and later, in [13, 14], where the string theory instantons were associated with the double points of of the complex curve. In this paper we study the quantum corrections to the leading NPC to the partition function, given by the pre-exponential factors. We calculate these factors using the matrix model description of the $c = 1$ string theory.

We perform the calculation by two different methods. The first method uses the fact that the free energy of the perturbed theory as a function of the cosmological constant μ and sine-Liouville coupling λ satisfies Toda partial differential equation. The second method is based on the formulation of the matrix model as a system of free fermions and consists in direct evaluation of the quasiclassical fermion wave functions. This method gives the answer for a general time-dependent background in terms of the complex curve associated with it.

We are using the chiral formalism in which the role of canonical coordinate and momentum are played by the left and right chiral combinations $x_\pm \sim x \pm p$. The advantage of this formalism is the exact bosonization of the fermion operators. Using the bosonization we interpret our results in terms of (target-space) CFT correlation functions. We show that the subleading corrections are related to the two-point correlation function of the bosonic field. Unlike the bosonization formulae for other matrix models for non-critical strings [37, 15, 38], here we have two chiral fields associated with the left and right moving tachyons. The resulting bosonic field theory can be viewed also as a boundary CFT, in which the boundary condition relates the left and right fields, see e.g. [39, 40] for similar statement for the normal matrix model. Since our bosonic field describes chiral excitations above the Fermi sea, its correlation functions do not have direct interpretation in terms of the usual FZZT branes. Nevertheless, the one-point functions, evaluated at the singular points of the complex curve, reproduce the disk partition function on ZZ branes, as is the case in the $c = 0$ matrix model [15, 16].

2. Time-dependent backgrounds in matrix quantum mechanics

2.1 The singlet sector of MQM as a system of free fermions

The matrix quantum mechanics can be viewed as reduction of a two-dimensional $\mathrm{U}(N)$ gauge theory to one dimension, and involves one gauge field $\mathbf{A}_{ij}$ and one scalar field $\mathbf{X}_{ij}$, both hermitian $N \times N$ matrices. The modern interpretation of the matrix path integral is as an effective open string theory on N D0 branes [6]–[9]. The matrix variable $\mathbf{X}$ describes the open-string tachyon field in presence of the N D0 branes, and only effect of the gauge field is that it projects onto the singlet sector. The theory is described by the action

$$\mathcal{S} = \int dt\, \mathrm{Tr} \left(\mathbf{P}\, \nabla_{\mathbf{A}} \mathbf{X} - \frac{1}{2}(\mathbf{P}^2 - \mathbf{X}^2) + \mu N \right), \tag{2.1}$$

where $\nabla_{\mathbf{A}} \mathbf{X} = \partial_t \mathbf{X} - i[\mathbf{A}, \mathbf{X}]$ is the covariant time derivative. The cosmological constant μ is introduced as a chemical potential for the size N of the matrices, which is treated as a dynamical variable.

The closed $c = 1$ string theory appears as a theory of collective excitations of the matrix variables. The momentum modes arise as collective excitations of the matrix $\mathbf{X}_{ij}$, while the winding modes are collective excitations of $\mathbf{A}_{ij}$. The vertex operators (1.3) can be represented in the matrix model by[3] [42]

$$\mathcal{T}_p \leftrightarrow \begin{cases} e^{-pt}\, \mathrm{Tr}\, \mathbf{X}_+^{|p|} & \text{if } p > 0 \\ e^{-pt}\, \mathrm{Tr}\, \mathbf{X}_-^{|p|} & \text{if } p < 0, \end{cases} \tag{2.2}$$

where X_+ and X_- are the chiral combinations of the matrix coordinate and momentum

$$\mathbf{X}_\pm = \frac{\mathbf{X} \pm \mathbf{P}}{\sqrt{2}} \,. \tag{2.3}$$

[3] When comparing the results in CFT and matrix model descriptions, we encounter the old problem of operator mixing [41]. This problem occurs because the correlation functions are integrated over the world sheet and the integrals have contributions from the coinciding points and a special prescription is needed to distinguish two tachyons with momenta p_1 and p_2 close to each other from a single tachyon with momentum $p_1 + p_2$. This ambiguity possibly leads to an analytic redefinition of the couplings when passing to the matrix model description.

The theory simplifies significantly if formulated directly in terms of the chiral variables $\mathbf{X}_\pm$ [36]. In the new variables the action (2.1) becomes

$$\mathcal{S} = \int dt \, \mathrm{Tr} \left(\mathbf{X}_+ \nabla_{\mathbf{A}} \mathbf{X}_- + \mathbf{X}_+ \mathbf{X}_- + \mu N \right) \tag{2.4}$$

so that the new hamiltonian is linear in the canonical coordinates and momenta.

In the singlet sector, characterized by absence of winding modes, the matrix model is described by a system of free fermions whose phase space is the spectral plane (x_+, x_-) of the two commuting matrices $\mathbf{X}_+$ and $\mathbf{X}_-$. The fermions are governed by a one-body hamiltonian

$$\hat{H}_0 = -\frac{1}{2}(\hat{x}_+ \hat{x}_- + \hat{x}_- \hat{x}_+) , \tag{2.5}$$

where

$$[\hat{x}_+, \hat{x}_-] = -i . \tag{2.6}$$

The one-particle wave functions in "x_+" and "x_-" representations are related by Fourier transformation $\hat{S}$:

$$\psi_-(x_-) = [\hat{S}\psi_+](x_-) \equiv \frac{1}{\sqrt{2\pi}} \int dx_+ e^{ix_+ x_-} \psi_+(x_+) . \tag{2.7}$$

The one-body hamiltonian has continuous spectrum and is diagonalized, in the x_+ and x_- representations, by the functions

$$\psi_\pm^E(x_\pm) = \frac{1}{\sqrt{2\pi}} x_\pm^{\pm iE - \frac{1}{2}} , \qquad E \in \mathbb{R} . \tag{2.8}$$

The wave functions (2.8) have branch points at $x_\pm = 0$ and therefore are defined unambiguously only for $x_\pm > 0$. It is useful to consider them either as multivalued meromorphic functions of the complex variables $x_\pm$, or as analytic functions of $\log x_\pm$.

We will restrict ourselves to the case of a theory with one Fermi sea on the right of the top of the potential, in which the wave functions are supported, up to exponentially small terms, by the positive axis. Therefore we will define ψ_+^E and ψ_-^E by (2.8) along the positive axis, and their values along the negative axis will be obtained by analytic continuation from the upper and lower half plane, respectively.[4] The wave functions (2.8) are orthonormal with respect to the scalar product

$$(f, g) = \int_0^\infty dx_\pm \overline{f(x_\pm)} \, g(x_\pm) , \tag{2.9}$$

where the integration with respect to the chiral phase space coordinates is performed only along the positive real axis, $x_\pm > 0$. The restriction of the phase space to $x_\pm > 0$ can be intuitively understood as follows. The wave function describes the system at given moment of time and its form for other moments is obtained by applying the evolution operator $e^{it\hat{H}}$. On the other hand, since the evolution operator shifts $\arg(x_\pm)$ by $\mp it$, the analytic continuation of the wave functions (2.8) off the positive axis can be viewed as the result of evolution in imaginary time direction.

[4] In a theory with two Fermi seas one should introduce a second set wave functions, which are obtained from $\psi_\pm^E$ by reflection $x_\pm \to -x_\pm$. For details see the appendix of [36].

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The Fourier transformation $\hat{S}$ defined by (2.7) gives the reflection part of the scattering operator, relating the incoming leftmovers and outgoing rightmovers. It acts diagonally on the wave functions (2.8):

$$\hat{S}\,\psi_{+}^{E} = e^{i\phi_0(E)}\,\psi_{-}^{E}\,, \tag{2.10}$$

with

$$e^{i\phi_0(E)} = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{\pi}{2}(E-i/2)}\,\Gamma\left(iE+\frac{1}{2}\right). \tag{2.11}$$

The reflection phase $\phi_0(E)$, is the same as the one calculated using the standard x-representation of the upside-down oscillator in [43, 2]. It has an exponentially small imaginary part

$$\operatorname{Im}\phi_0(-\mu) = \frac{1}{2}\log(1+e^{-2\pi\mu})\,, \tag{2.12}$$

which determines the flow of particles under the potential barrier.

2.2 Partition function and density of states

The string theory compactified at time interval $\beta = 2\pi R$ is described by the grand canonical ensemble of fermions at finite temperature $1/\beta$ and chemical potential μ. If $\mathcal{Z}(\mu)$ is the partition function of the ensemble of fermions, then the free energy $\mathcal{F} = \log\mathcal{Z}$ is given by

$$\mathcal{F}(\mu) = \int_{-\infty}^{\infty} dE\,\rho(E)\log\left(1+e^{-\beta(\mu+E)}\right), \tag{2.13}$$

where $\rho(E)$ is the density of states related to the phase $\phi(E)$ of the fermion scattering by

$$\rho(E) = \frac{\log\Lambda}{2\pi} - \frac{1}{2\pi}\frac{d\phi(E)}{dE}\,. \tag{2.14}$$

The non-trivial part of the density adds to a constant cut-off dependent term, where the cut-off Λ is introduced as the volume of the (x_+, x_-) phase space. We remind the derivation of this relation in appendix A, where we also recall the derivation of the relation between the free energy and the scattering phase

$$2\sin\frac{\partial_\mu}{2R}\cdot\mathcal{F}(\mu) = \phi(-\mu)\,. \tag{2.15}$$

As we know from [3], this last identity allows to evaluate all tachyon correlation functions in the compactified theory, once their counterparts in the theory with $R=\infty$ are known.

The identities (2.14) and (2.15) hold for any Fermi system at finite temperature and, in particular, after an arbitrary time-dependent perturbation that preserves the singlet sector. Thus all the information about the system is contained in the phase of fermionic scattering $\phi(E)$ as a function of the coupling constants $\{t_k\}$ associated with the tachyon operators (2.2).

Let us recall the explicit expressions for the stationary background, where all $t_k = 0$. Then the phase $\phi_0(E) = \phi(E)_{\{t_k=0\}}$ is given by (2.11) and the free energy itself is calculated by inverting the finite-difference operator in (2.15). The latter is diagonalized by the Fourier transformation

$$\phi_0(-\mu) = -\frac{i}{2}\int_{\frac{1}{\Lambda}}^{\infty}\frac{ds}{s}\frac{e^{i\mu s}}{\sinh\frac{s}{2}}\,, \tag{2.16}$$

where we reintroduced the cut-off Λ. Then (2.15) yields[5]

$$\mathcal{F}(\mu)_{\{t_k=0\}} = -\frac{1}{4}\int_{\frac{1}{\Lambda}}^{\infty}\frac{ds}{s}\frac{e^{i\mu s}}{\sinh\frac{s}{2}\,\sinh\frac{s}{2R}}\,. \tag{2.17}$$

The real part of this integral is the well-known integral representation for the partition function of the compactified $c = 1$ string theory [3]. Its genus expansion (which is also the expansion in $1/\mu$) reads

$$\mathcal{F}_{\rm pert}(\mu)_{\{t_k=0\}} = -\frac{R}{2}\mu^2\log\frac{\mu}{\Lambda} - \frac{R+\frac{1}{R}}{24}\log\frac{\mu}{\Lambda} + R\sum_{h=2}^{\infty}\mu^{2-2h}c_h(R)\,, \tag{2.18}$$

where the genus h term $c_h(R)$ is a known polynomial in $1/R$.

In the Fermi system, the tachyons are represented by collective excitations of the Fermi liquide [45, 2]. A general time-dependent background, corresponding to the tachyon deformation (1.5) of the world-sheet CFT, can be introduced by changing the state of the fermi system in the infinite past and in the infinite future, in the spirit of the Polchinski's derivation of the tree-level tachyon S-matrix. This amounts to modifying the asymptotics at infinity of the fermion eigenfunctions (2.8) in x_+ and x_- representations. The spectrum of the tachyons should be of the form (1.4) in order to have periodicity in $\beta = 2\pi R$ in the imaginary time direction. As we know, such deformations of MQM are described by Toda hierarchy [46, 34, 36]. The commuting Toda flows can be formulated either as an hierarchy of PDE with respect to the 'times' $t_{\pm k}$, or in terms of a pair of Lax operators satisfying a string equation [47]. Below we remind the necessary facts about the two approaches.

2.3 Time-dependent backgrounds via Toda hierarchy

The partition function of the perturbed theory is a τ-function of Toda hierarchy and satisfies a hierarchy of PDE with respect to the couplings $t_{\pm k}$. The first of them is the Toda equation, which is sufficient to describe sine-Liouville perturbation $t_1 = t_{-1} = \lambda$. Written for the free energy $\mathcal{F} = \log\mathcal{Z}$, this equation has the form

$$\frac{1}{4}\lambda^{-1}\partial_\lambda\lambda\partial_\lambda\mathcal{F}(\mu,\lambda) + \exp\left[-4\sin^2\left(\frac{\partial_\mu}{2R}\right)\mathcal{F}(\mu,\lambda)\right] = 1 \tag{2.19}$$

and should be solved with initial condition provided by (2.18).

Equation (2.19) defines the flow between the critical points $\lambda = 0$ and $\mu = 0$ of the world-sheet CFT. When λ is large, so that sine-Liouville term sets the scale, the cosmological term can be considered as a perturbation. In this case it is convenient to introduce the following variables[6]

$$y = \mu\xi\,, \qquad \xi = \left(\lambda^2\,\frac{1-R}{R^3}\right)^{-\frac{R}{2R-1}}\,. \tag{2.20}$$

[5]Up to cut-off-dependent terms this integral gives the logarithm of the Barnes [44] Gamma function: $\mathcal{F}(\mu,0) = \log\Gamma_2(\frac{1+R}{2} - i\mu|1,R)$.

[6]This definition is adjusted to the case $R < 1$; the case $R > 1$ can be treated similarly.

The variable y is a dimensionless parameter and $\xi \sim g_{\rm str}$. Therefore, the genus expansion of the free energy is an expansion in ξ with y-dependent coefficients. For each coefficient in the expansion the PDF (2.19) yields an ordinary differential equation [34]. Below we will only need the genus zero contribution $\mathcal{F}_0$ to the free energy. In the variables (2.20) it has the form

$$\mathcal{F}_0(\mu,\lambda) = \frac{1}{\xi^2}\left[\frac{R}{2}y^2\log\xi + f_0(y)\right], \tag{2.21}$$

the differential equation satisfied by f_0 can be integrated to an algebric equation for the function

$$X(y) \equiv \partial_y^2 f_0(y)\,, \tag{2.22}$$

namely

$$y = e^{-\frac{1}{R}X(y)} - e^{-\frac{1-R}{R^2}X(y)}\,. \tag{2.23}$$

2.4 Time-dependent backgrounds via fermion wave functions

As we mentioned, in the fermionic picture one can introduce sources for incoming and outgoing tachyons by changing the asymptotics of the one-particle wave functions. The perturbed one-fermion wave functions are obtained from the bare wave functions (2.8) by multiplying with a coordinate-dependent phase factor

$$\Psi^E_\pm(x_\pm) = e^{\mp i\varphi_\pm(x_\pm;E)}\,\psi^E_\pm(x_\pm)\,. \tag{2.24}$$

The phases can be written as a sum of three terms

$$\varphi_\pm(x_\pm;E) = V_\pm(x_\pm) + \frac{1}{2}\phi(E) + v_\pm(x_\pm;E)\,, \tag{2.25}$$

where $V_\pm$ vanishes for $x_\pm = 0$, $v_\pm$ vanishes for large $x_\pm \to \infty$, and $\phi(E)$ is a constant. The deformations (1.5) of the world sheet CFT correspond, up to a possible analytic redefinition of the couplings, to the choice

$$V_\pm(x_+) = \sum_{k\geq 1} t_{\pm k}\, x_\pm^{k/R}\,. \tag{2.26}$$

The constant mode $\phi(E)$ and the pieces $v_\pm(x_\pm;E)$ depend implicitly on $t_{\pm n}$. They are determined from the requirement that the left and right fermions are related by the Fourier transformation (2.7):

$$\hat{S}\Psi^E_+ = \Psi^E_-\,, \qquad \hat{S}^{-1}\Psi^E_- = \Psi^E_+ \tag{2.27}$$

where we absorbed the reflection phase factors $e^{i\phi(E)}$ in the definition of the deformed wave functions.

For large negative energy E, the compatibility of the saddle-point equations for the two integrals in (2.27) gives

$$\begin{aligned} x_+x_- &= \frac{1}{R}\sum_{k\geq 1} kt_k\, x_+^{k/R} - E + \frac{1}{R}\sum_{k\geq 1} v_k(E)\, x_+^{-k/R} \\ &= \frac{1}{R}\sum_{k\geq 1} kt_{-k}\, x_-^{k/R} - E + \frac{1}{R}\sum_{k\geq 1} v_{-k}(E)\, x_-^{-k/R}\,. \end{aligned} \tag{2.28}$$

This means that the two equations (2.28) define two functions $x_\pm = X_\pm(x_\mp)$ which are inverse to each other:

$$X_+(X_-(x_+)) = x_+ \,. \tag{2.29}$$

The functions $X_\pm(x_\mp)$, taken at the Fermi level $E_F = -\mu$, can be considered as the gluing functions for an analytic curve $\mathcal{M}$ in $\mathbb{C}^2$. This curve yields all the information for the given background. Its real section determines the shape of the Fermi sea in the quasiclassical limit $\mu \to \infty$. The functions $X_\pm(x_\mp)$ are in general multi-valued, but there is always a global parameter, the "proper time" variable τ, such that the solution of (2.28) is given by $x_\pm = x_\pm(\tau)$. If all couplings $t_{\pm k}$ with $k > k_{\max}$ vanish, then the solution is of the form [47]

$$x_\pm(\tau) = e^{\pm\tau - \frac{1}{2R}\chi}\left(1 + \sum_{k=1}^{k_{\max}} a_{\pm k}\, e^{\mp\frac{k}{R}\tau}\right), \tag{2.30}$$

where the quantity χ is related to the constant mode of the phase (2.25) via

$$\chi = -R\frac{\partial}{\partial_E}\phi \,. \tag{2.31}$$

The coefficients $a_{\pm k}$ and χ can be found by substituting (2.30) into (2.28) and comparing coefficients in front of $e^{\pm\frac{k}{R}\tau}$. For example, in the case of sine-Liouville deformation, i.e. when only the first coupling constants are nonzero, this procedure gives

$$\mu e^{\frac{1}{R}\chi} - \frac{1}{R^2}\left(1 - \frac{1}{R}\right)\lambda^2 e^{\frac{2R-1}{R^2}\chi} = 1 \,, \qquad a_{\pm 1} = \frac{\lambda}{R}\, e^{\frac{R-1/2}{R^2}\chi} \,. \tag{2.32}$$

Taking into account that $\partial_\mu^2 \mathcal{F}_0 = \chi$, it is easy to show that this equation is equivalent to the solution (2.23) of Toda equation. The solution beyond the tree level can be found using the full Toda integrable structure.

The parametrization (2.30) can be thought of as a canonical transformation relating the phase space coordinates (x_-, x_+) and (τ, E), where the proper time τ appears as the variable canonically conjugated to the energy E:

$$\{x_-, x_+\} = 1 \quad \Leftrightarrow \quad \{\tau, E\} = 1 \,. \tag{2.33}$$

The origin of the last relation can be traced to the Lax representation of the operators $x_\pm$ in the basis of the deformed wave functions, where the parameter $\omega = e^\tau$ arises as the classical limit of the shift operator $\hat{\omega} = e^{-i\partial_E}$ [36, 47].

3. NPC in sine-Liouville string theory from Toda equation

In this section we restrict ourselves to the sine-Liouville deformation with coupling λ, in which case the free energy can be determined from Toda equation (2.19). The initial condition is provided by the free energy of the non-deformed theory which is given by the integral (2.17). Before studying the theory with the sine-Liouville deformation, let us remind the exact expression for the NPC in the non-deformed $c = 1$ string theory following from the integral representation (2.17). The integral has a small imaginary part, which describes the flow of eigenvalues beyond the top of the inverse oscillator potential. It can be evaluated by extending the contour to the whole real axis and taking the residues at the two series of poles, $s_n = 2i\pi n$ and $s_n = 2i\pi Rn$:

$$\mathcal{F}_{\rm np}(\mu)_{\{t_k=0\}} = i\sum_n \frac{e^{-2\pi n\mu}}{4n(-1)^n \sin\frac{\pi n}{R}} + i\sum_n \frac{e^{-2\pi Rn\mu}}{4n(-1)^n \sin(\pi Rn)}\,. \tag{3.1}$$

This expression is of course compatible with (2.12) and (2.15). We see that there are two types of NPC, which have their origin in the two kinds of branes in the $c = 1$ string theory [10]. The first type is due to the D-instanton with Dirichlet boundary conditions on both matter and Liouville fields, whereas the second one is due to a D0-brane with Neumann boundary condition for the matter field. In both cases the pre-exponential factors do not depend on μ and therefore scale like $g_{\rm str}^0$.

The two types of NPC behave very differently with sine-Liouville coupling λ. The NPC of the type $e^{-2\pi Rn\mu}$ do not flow with λ and are always given by the second term of (3.1). Indeed, given a solution of Toda equation (2.19), by adding a linear combination of exponents $e^{-2\pi R\mu k}$ with λ-independent coefficients one obtaines another solution.

In contrast, the NPC given at $\lambda = 0$ by the first term of (3.1), evolve non-trivially with λ. The leading exponential contributions of this series were analyzed in [10, 11] for deformation by vortices. From these works we know[7] that the leading non-perturbative effects have the form

$$\varepsilon_n(\mu,\lambda) \sim e^{-2g_n(y)/\xi}\,, \tag{3.2}$$

where

$$g_n(y) = y\theta_n(y) + \frac{1}{\sqrt{\alpha}}\, e^{-\frac{X(y)}{2R^2}} \sin\frac{\theta_n(y)}{R}\,, \qquad \alpha = \frac{1-R}{R(2R-1)^2} \tag{3.3}$$

and $\theta_n(y) = \partial_y g_n$ is found as a solution of the algebraic equation

$$\sin\theta_n = \left(\frac{1}{R} - 1\right)^{-\frac{1}{2}} e^{\frac{2R-1}{2R^2}X(y)} \sin\left(\frac{1-R}{R}\,\theta_n\right). \tag{3.4}$$

The different solutions of (3.4) are labeled by the integer n in such way that $\theta_n \to \pi n$ when $\lambda \to 0$.

Now we would like to find the subleading order, i.e. the factor in front of the exponential in (3.2). Following the procedure suggested in [10], we take two solutions $\mathcal{F}$ and $\tilde{\mathcal{F}}$ of (2.19) that differ by the exponentially small quantity $\varepsilon = \tilde{\mathcal{F}} - \mathcal{F}$. Then ε must satisfy the linearized equation

$$\frac{1}{4}\lambda^{-1}\partial_\lambda \lambda \partial_\lambda \varepsilon(\mu,\lambda) - 4\, e^{-\frac{1}{R^2}\partial_\mu^2 \mathcal{F}_0(\mu,\lambda)} \sin^2\left(\frac{\partial_\mu}{2R}\right) \varepsilon(\mu,\lambda) = 0\,, \tag{3.5}$$

where in the exponent in the second term we approximated

$$4R^2 \sin^2\left(\frac{\partial_\mu}{2R}\right) \mathcal{F}(\mu,\lambda) \simeq \partial_\mu^2 \mathcal{F}_0 = R\log\xi + X(y)\,. \tag{3.6}$$

[7]The expressions obtained in [10, 11] should be used after the T-duality transformation $\xi \to \xi/R,\ y \to y$.

This approximation is correct for our purpose since the subleading non-perturbative contribution is of order $O(g_{\rm str})$ with respect to the leading one, whereas the perturbative genus expansion goes in powers of $g_{\rm str}^2$. Finally, changing the variables from (λ,μ) to (ξ,y) we write equation (3.5) as

$$\alpha\xi^2(y\partial_y+\xi\partial_\xi)^2\varepsilon(\xi,y)=4\,e^{-\frac{X(y)}{R^2}}\sin^2\left(\frac{\xi}{2R}\,\partial_y\right)\varepsilon(\xi,y)\,. \tag{3.7}$$

This equation is solved by a refinement of the Ansatz (3.2):

$$\varepsilon(\mu,y)\simeq\ A(\xi,y)\,e^{-2g(y)/\xi}\,,\qquad A(\xi,y)=[\xi a(y)]^b\,, \tag{3.8}$$

where b may depend on y. We omit the index n since it does not appear explicitly in the equations. The details of the solution are presented in appendix B. The result is that $b=\text{const}$ and $a(y)$ is given by (B.9).

The constant b, which gives the power of $g_{\rm str}$ in the pre-exponential factor, is not fixed by Toda equation. It looks like an integration constant. Therefore, the first idea is that it can be fixed from the initial condition (3.1) at $\lambda=0$, which implies that it should vanish. On the other hand, let us consider the limit of small λ. In this limit y is large and

$$\theta_n(y)\approx\pi n+\left(\frac{1}{R}-1\right)^{-\frac12}\sin\frac{\pi n}{R}\,y^{-\frac{2R-1}{2R}}\,. \tag{3.9}$$

This leads to the asymptotics

$$A(\xi,y)\sim\left(\frac{\xi}{y}\right)^b y^{\frac{2R-1}{4R}}\sim\lambda^{-1/2}\,\mu^{\frac{2R-1}{4R}-b}\,. \tag{3.10}$$

We observe that the subleading contribution does not have smooth small λ limit. Therefore, the parameter b is not fixed by this approach. As we will see in the next section, it should be fixed to give the same power of $g_{\rm str}$ as in $c<1$ string theories,

$$b=\frac{1}{2}\,, \tag{3.11}$$

so that the full prefactor is given by the following function

$$A(\xi,y)=C\left(e^{-\frac{1}{R}\chi}\,\sin^2\theta\,\sin^2\frac{\theta}{R}\,\left[\left(\frac{1}{R}-1\right)\cot\left(\frac{1-R}{R}\,\theta\right)-\cot\theta\right]\right)^{-1/2}\,. \tag{3.12}$$

The overall coefficient C remains undetermined. Thus, we see that sine-Liouville deformation changes drastically the behavior of the non-perturbative corrections in the subleading order: the power of $g_{\rm str}\sim\xi$ changes discontinuously from 0 to 1/2. This means that the limits $\Lambda\to\infty$ and $\lambda\to 0$ do not commute. As we will see later, the change of the behavior of the non-perturbative corrections can be explained with the fact that the tachyon perturbations break the time translation symmetry of the theory.

A nice consistency check of the result (3.12) is to reproduce the non-perturbative corrections to the free energy of the pure gravity in the limit where the couplings approach the $c=0$ critical point. We discuss this limit in appendix C where we show that the result (3.12) does reproduce the correct critical behavior.

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4. NPC from the quasiclassical wave functions

The approach based on Toda equation is conceptually straightforward but does not allow to fix the solution completely. Alternatively one can exploit the fermionic formulation to evaluate the non-perturbative corrections to the fermion reflection phase and then use the relation (2.15) to find those for the partition function. Moreover, this second approach allows to generalize the results obtained in the previous section to the case of a deformation with arbitrary number of non-vanishing couplings. Before considering the general case, we will illustrate the method for the stationary background, where the exact form of the non-perturbative corrections is known.

4.1 The case of a stationary background

Although the exact expression for the reflection phase is known, we follow here another approach which works only quasiclassically but can be generalized to other situations. For this we use the following matrix element of the scattering operator (2.10)

$$\frac{1}{\sqrt{2\pi}}\iint_0^\infty dx_+ dx_- \, \overline{\psi_-^E(x_-)}\, e^{ix_+x_-}\, \psi_+^{E'}(x_+) = e^{i\phi_0(E)}\delta(E-E')\,, \tag{4.1}$$

where we used the orthonormality of the wave functions (2.8). Substituting the explicit expression for the wave functions (2.8) and introducing a cut-off Λ equal to the volume of the phase space, we write the diagonal matrix element as

$$\frac{1}{(2\pi)^{3/2}}\iint_0^{\sqrt{\Lambda}} \frac{dx_+ dx_-}{\sqrt{x_+x_-}}\, e^{i\left(x_+x_- + E\log(x_+x_-)\right)} = e^{i\phi_0(E)}\rho_0(E)\,, \tag{4.2}$$

where $\rho_0(E) \approx \frac{1}{2\pi}\log(-\Lambda/E)$ is the density of states corresponding to the cut-off Λ. In the following discussion we can safely approximate $\rho_0(E)$ by $\frac{1}{2\pi}\log\Lambda$.

We would like to evaluate the integral (4.2) by the saddle point method. The two saddle point equations associated with the variables x_+ and x_- actually coincide and both give the equation for the classical fermion phase-space trajectory with energy E:

$$x_+x_- = -E\,. \tag{4.3}$$

The fact that the two equations coincide means that we do not have isolated saddle points but 'saddle contours' that go along the flat direction. If we change the variables to

$$x_\pm(\tau) = \sqrt{\epsilon}\, e^{\pm\tau}\,, \tag{4.4}$$

then the flat direction is along the 'proper time' τ. The cut-off prescription $0 < x_\pm < \sqrt{\Lambda}$ then restricts the τ-integration in (4.2) to the interval

$$-\frac{1}{2}\log\frac{\Lambda}{\epsilon} < \tau < \frac{1}{2}\log\frac{\Lambda}{\epsilon} \tag{4.5}$$

and the integral in τ gives the factor $\log\frac{\Lambda}{\epsilon} \approx 2\pi\rho_0(-\epsilon)$. The remaining integral in ϵ is

$$\int_0^\Lambda \frac{d\epsilon}{\sqrt{2\pi\epsilon}}\, e^{i(E\log\epsilon+\epsilon)} = e^{i\phi_0(E)}\,. \tag{4.6}$$

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The saddle point is at $\epsilon = -E$ and the gaussian fluctuations cancel $\sqrt{2\pi\epsilon}$ in the denominator. This gives the genus-zero perturbative contribution for the reflection phase:

$$\phi_0(E) \approx E \log(-E) - E \,. \tag{4.7}$$

The appearance of the non-perturbative corrections is related to the fact that the phase of the integrand in (4.2) is a multivalued function of ϵ. As a consequence, there is an infinite number of saddles at

$$\log \epsilon_n = \log(-E) - 2\pi i n \tag{4.8}$$

with n integer. The saddles with $n > 0$ can be connected by constant phase contours to the dominant saddle and thus are potentially relevant. The integral along a contour that passes through the n-th subdominant saddle contributes a factor $(-1)^n e^{2\pi n E}\, e^{i(E\log(-E)-E)}$, where $(-1)^n$ comes from the pre-factor in (4.6). Taking into account the contribution of all saddles, one finds for the Fermi level $E = -\mu$

$$e^{i\phi_0(-\mu)} \approx e^{-i(\mu\log\mu - \mu)} \left(1 + \sum_{n\in\mathbb{N}} c_n \, (-1)^n \, e^{-2\pi n\mu} \right), \tag{4.9}$$

where the coefficients c_n depend on the choice of the integration contour. This is in agreement with the expression that follows from the exact answer (2.12),

$$\phi_0(-\mu) \approx -\mu\log\mu + \mu - i \sum_n \frac{1}{2n} \, (-1)^n \, e^{-2\pi n\mu} \,. \tag{4.10}$$

The coefficient $c_1 = 1/2$ in the contribution of the first subdominant saddle can be explained by the fact that the contour turns at $\pi/2$ after reaching the saddle [48]. The choice of the integration contours and the evaluation of the contributions of the subdominant saddles is a delicate problem and we will not discuss it here.

Instead, we will try to understand the geometrical meaning of the saddles (4.8) in terms of the complex curve defined by the saddle point equation (4.3) at the Fermi level $E_F = -\mu$,

$$x_+ x_- = \mu \,. \tag{4.11}$$

As a real manifold the curve represents a hyperboloid, depicted in figure 1a. It is characterized by two non-contractible cycles, the compact A-cycle, given by the section $\bar{x}_+ = x_-$, and the non-compact B-cycle given by the connected component of the section $\bar{x}_\pm = x_\pm$ with $x_\pm > 0$, which coincides with the profile of the Fermi sea. The B-cycle connects the two infinite points $\infty_- = \{x_- = \infty\}$ with $\infty_+ = \{x_+ = \infty\}$. A global parametrization of $\mathcal{M}_0$ is given by the map

$$x_\pm = \sqrt{\mu}\, e^{\pm\tau} \tag{4.12}$$

to the strip $|\operatorname{Im}\tau| \le \pi$.

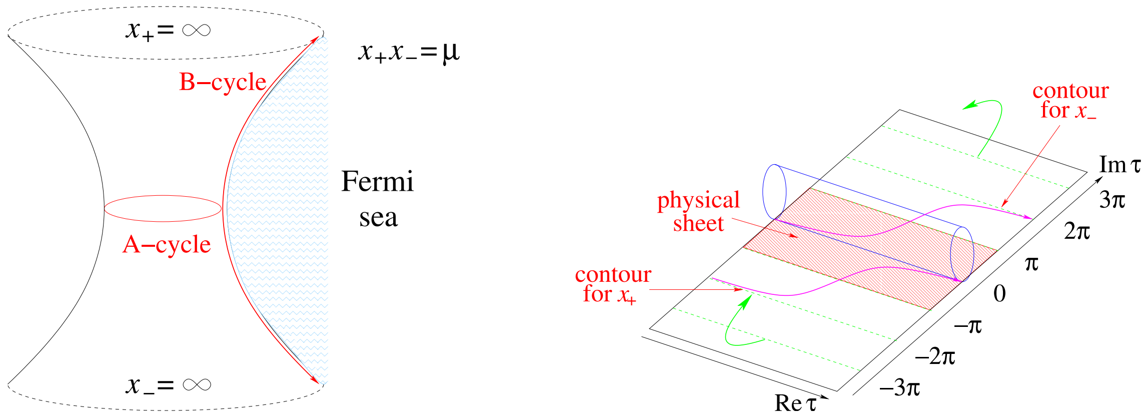

Figure 1: The Riemann surface of the non-deformed theory (one dimension is suppressed). The "saddle contours" are given by the B-cycle and n copies of the A-cycle. The second picture represents the covering of the hyperboloid by the complex τ-plane and non-minimal saddle contours for $x_\pm(\tau)$ in the universal cover.

However, because of the multi-valuedness of the integrand in (4.2), the solutions of the saddle point equation are actually described by the universal cover $\overline{\mathcal{M}_0}$ of the curve (4.11), globally parameterized by the whole complex τ-plane. For the minimal saddle, $n = 0$, the integration contours for dx_+ and dx_+ belong to the same sheet of the universal cover $\mathcal{M}_0$ and coincide with the cycle B on the hyperboloid, which is parameterized by the real axis in the τ-plane.

In contrast, the non-minimal saddles, $n \geq 1$, are described by two *different* contours on the universal cover $\overline{\mathcal{M}_0}$: one of the contours describes $x_+(\tau)$ and the other one gives $x_-(\tau)$. In the parametrization (4.12), they are represented as contours joining $-\infty - 2\pi i n$ to ∞ and $-\infty$ to $\infty + 2\pi i n$, respectively, as the ones drown in figure 1*b*. These conditions ensure that $\mathrm{Im}(\log x_\pm) = 0$ in the asymptotic regions of large $x_\pm$. The two contours have the same projection in $\mathcal{M}_0$ where they follow the cycle B and at some point wind n times around the cycle A, as it is shown on fig 1a. But since x_+ and x_- follow the different contours in the universal cover, the integrand in (4.2) along such a contour yields an additional factor $(-1)^n e^{-2\pi n\mu}$.

In this way, the non-minimal saddles are assiciated with 'double contours' on the hyperboloid (4.11), rather than with double points. The double points appear in the complex curve for the resolvent, $y = w(x)$, which is also the $c \to 1$ limit of the FZZT curves of minimal string theories. The curve for the resolvent is parametrized as

$$x = \sqrt{2\mu}\cosh\tau\,, \qquad w = -\frac{\sqrt{2\mu}}{\pi}\,\tau\,\sinh\tau \tag{4.13}$$

and represents a $\mathbb{Z}_2$ orbifold of the universal cover of the hyperboloid (4.11), obtained by identifying the points τ and $-\tau$. There are two points on the curve $(\pm\sqrt{2\mu}, 0)$, which are also the positions of the branch points of $y = w(x)$, which are images of infinitely many points on the τ-plane $\tau = i\pi n$. They appear as the (degenerate) limit of the double points of the minimal string curves.

4.2 Quasiclassical wave functions in the general case

Now we will apply the same method to calculate the NPC to the scattering phase in the case of a general tachyon deformation described in section 2.4. The first step of the calculation consists in the evaluation of the quasiclassical asymptotics of the fermion wave functions.

First of all, we need the classical limit of the phases $\varphi_\pm(x_\pm, E)$ in (2.24), which is determined by the compatibility of two equations (2.28). The integration of these equations gives the following representation

$$\varphi_\pm(x_\pm) = \int_\infty^{x_\pm} X_\mp(x'_\pm)\, dx'_\pm + E\log x_\pm + \phi_\pm\,, \tag{4.14}$$

where the functions $x_\pm = X_\pm(x_\mp)$ are the classical fermion trajectories defined in parametric form by (2.30) and $\phi_\pm$ are integration constants. As a result, in this approximation the wave functions take the form

$$\Psi^E_\pm(x_\pm) \approx B_\pm(x_\pm)\, e^{\mp i\phi_\pm} e^{\mp i\int_\infty^{x_\pm} X_\mp(x'_\pm)\, dx'_\pm}\,, \tag{4.15}$$

where $B_\pm$ are factors coming from the subleading order in the Planck constant. These factors and the zero mode $\phi = \phi_+ + \phi_-$ can be fixed from the normalization condition similar to (4.1).[8]

With the cut-off Λ introduced as in the previous subsection, the wave functions satisfy the normalization condition

$$\int_0^{\sqrt{\Lambda}} dx_\pm\, \overline{\Psi^E_\pm(x_\pm)}\ \Psi^E_\pm(x_\pm) = \frac{1}{2\pi}\log\Lambda\,. \tag{4.16}$$

A second condition follows from the action of the scattering operator, that is the Fourier transformation. Since we absorbed the scattering phase into the wave functions, the Fourier transformation acts as the identity operator, which leads, together with (4.16), to

$$\frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{\Lambda}} dx_+ \int_0^{\sqrt{\Lambda}} dx_-\ \overline{\Psi^E_-(x_-)}\, e^{ix_+x_-}\ \Psi^E_+(x_+) = \rho(E)\,. \tag{4.17}$$

Substituting the asymptotic form (4.15) and keeping only the constant cut-off dependent piece of the density (2.14), one writes the two conditions as

$$\int_0^{\sqrt{\Lambda}} dx_\pm\, |B_\pm|^2 = \frac{1}{2\pi}\log\Lambda\,, \tag{4.18}$$

$$\int_0^{\sqrt{\Lambda}} dx_+ \int_0^{\sqrt{\Lambda}} dx_-\ \bar{B}_- B_+\, e^{iS(x_+,x_-)} = \frac{e^{i\phi}}{\sqrt{2\pi}}\log\Lambda\,, \tag{4.19}$$

where we introduced the effective action

$$S(x_+,x_-) = x_+x_- - \int_\infty^{x_+} X_-(x'_+)\, dx'_+ - \int_\infty^{x_-} X_+(x'_-)\, dx'_-\,. \tag{4.20}$$

[8]Note that the deformed wave functions are not orthogonal anymore.

The integral in the second condition can be evaluated by the saddle point method. The property (2.29), following from the compatibility of two equations (2.28), means that the leading contribution to the integral is again associated with a one-dimensional saddle contour $\gamma_{\rm F}$ in the (x_+, x_-) plane, defined by the functions $x_\pm = X_\pm(x_\mp)$ and going from $x_- = \infty$ to $x_+ = \infty$. For $E = -\mu$ the saddle contour defines the profile of the Fermi sea.

To evaluate the gaussian fluctuations in the transversal direction to the saddle contour, it is convenient to change variables from (x_+, x_-) to $(\tau, -\epsilon)$, where τ is the same as in (2.30) and the variable ϵ parameterizes the transversal direction. The equation of the saddle contour in the new variables is $\epsilon = -E$. For the quadratic form $\delta_\epsilon^2 S(x_+, x_-)$ one finds, using (2.33),

$$
\begin{aligned}
\delta_\epsilon^2 S(x_+, x_-) &= \left(2\partial_\epsilon x_+ \partial_\epsilon x_- - \frac{dx_-}{dx_+}\left(\partial_\epsilon x_+\right)^2 - \frac{dx_+}{dx_-}\left(\partial_\epsilon x_-\right)^2\right)\frac{(\delta\epsilon)^2}{2} \\
&= \left(\frac{\partial_\epsilon x_+}{\partial_\tau x_+} - \frac{\partial_\epsilon x_-}{\partial_\tau x_-}\right)\frac{(\delta\epsilon)^2}{2} \\
&= -\frac{1}{\partial_\tau x_+ \partial_\tau x_-}\frac{(\delta\epsilon)^2}{2}\,.
\end{aligned}
\tag{4.21}
$$

As a result, the condition (4.19) is written as an integral along the saddle contour $\gamma_{\rm F}$:

$$
e^{i\int_{\gamma_{\rm F}} x_- dx_+} \int_{\gamma_{\rm F}} d\tau\ \bar{B}_- B_+ \left[-\partial_\tau x_+ \partial_\tau x_-\right]^{1/2} = \frac{e^{i\phi}}{2\pi}\log\Lambda\,. \tag{4.22}
$$

The two conditions (4.18) and (4.22) are satisfied if

$$
\phi = \phi_{\rm pert}(E) = \int_{\gamma_{\rm F}} x_-\, dx_+\,, \qquad B_\pm = \frac{1}{\sqrt{\pm 2\pi \partial_\tau x_\pm}}\,. \tag{4.23}
$$

In particular, taking into account the relation (2.15), this result gives the expression of the μ-derivative of the perturbative free energy $\mathcal{F}_{\rm pert}$ as an integral over the B-cycle on the complex curve (2.30) [49]. Knowing the wave functions (4.15), one can also reconstruct the quasiclassical wave function in the x-representation. We refer to appendix D for details.

4.3 Instantons as double points of the complex curve

In order to find the non-perturbative corrections for the free energy it is sufficient to calculate those for the scattering phase $\phi(E)$. This can be done by taking into account also the non-minimal saddles of the integral (4.19). We saw in section 4.1 that if there is no tachyon potential, the non-minimal saddles are given by pairs of one-dimensional contours living on different sheets of the universal cover of the hyperboloid $x_+x_- = \mu$, but having the same image in $\mathbb{C}^2$. The collective coordinate along these saddles is the Minkowski time, since the background is time-independent. In presence of tachyon potential there is no time translational invariance. As a consequence, the minimal saddle contour will evolve with time and the non-minimal saddles will be given by isolated saddle points, or double points of the deformed complex curve, to be described below. The meaning of these double points is essentially the same as in the minimal string theories [12, 14].

Again, the saddle-point equations $x_+ = X_+(x_-)$ and $x_- = X_-(x_+)$ define a complex curve $\mathcal{M}$ in $\mathbb{C}^2$, which is a deformation of the universal cover of the hyperboloid $x_+x_- = \mu$. The Riemann surfaces of the functions X_+ and X_- are the projections of the curve $\mathcal{M}$ to the x_- and x_+ complex planes. Due to the multi-valuedness of the functions $X_+(x_-)$ and $X_-(x_+)$, equation (2.29) has also particular solutions describing isolated double points, where the complex curve touches itself [13].

The double points can be classified most easily by using the uniformization parameter τ, defined in equation (2.30). They represent pairs of points $\tau' \neq \tau''$ in the complex τ-plane such that

$$x_+(\tau') = x_+(\tau'') \qquad \text{and} \qquad x_-(\tau') = x_-(\tau'') \,. \tag{4.24}$$

Each double point is by construction a solution of (2.29) and as such represents a saddle point for the action (4.20).

Let us restrict ourselves to the case $t_k = t_{-k}$ for all k, which is simpler to analyze. Then $a_k = a_{-k}$ in (2.30) and the set of double points is given by the pairs $(\tau' = -i\theta_n, \tau'' = i\theta_n)$, where θ_n are determined by

$$\sin(\theta_n) = \sum_{k=1}^{k_{\max}} a_k \sin\left(\frac{k-R}{R}\,\theta_n\right), \qquad \theta_n \to \pi n \quad \text{when all } t_k \to 0\,, \tag{4.25}$$

As τ' and τ'' are pure imaginary and complex conjugated, $x_+(\pm i\theta_n) = x_-(\pm i\theta_n)$ are both real. For the sine-Liouville deformation the discrete set of parameters θ_n is determined by the equation

$$\sin(\theta_n) = a(y)\sin\left(\frac{1-R}{R}\,\theta_n\right) \tag{4.26}$$

with $a(y)$ from (2.32). This equation is the same as (3.4) and therefore the parameters θ_n coincide with those defined in section 3.

Let us calculate the contribution of the n-th saddle point to the normalization integral (4.19). The leading contribution is given by the value of the action (4.20), which depends on the integration contours for dx_+ and dx_- along the complex curve. The two contours can be parameterized in the τ-plane by the intervals

$$\begin{aligned} \tau \in\ & (\infty, 0) \cup (0, -i\theta_n) \qquad \text{for } x_+(\tau)\,, \\ \tau \in\ & (-\infty, 0) \cup (0, i\theta_n) \qquad \text{for } x_-(\tau)\,. \end{aligned} \tag{4.27}$$

As a result, the action is given by $S_{\text{pert}} + iS_n$,

$$S_{\text{pert}} = \int_{\gamma_{\text{F}}} x_-\, dx_+\,, \qquad S_n = i\oint_{\gamma_n} x_-\, dx_+\,, \tag{4.28}$$

where the contours γ_{F} and γ_n are respectively the images of the intervals $(-\infty, \infty)$ and $(i\theta_n, -i\theta_n)$ in the τ-plane.

The first term $S_{\rm pert}$ gives, according to eq. (4.23), the leading perturbative approximation to the scattering phase. Its integration contour is the B-cycle connecting the two punctures at $x_\pm = \infty$. On the other hand, the image of the integration contour γ_n for the second term S_n is a closed loop in $\mathbb{C}^2$, because it ends at the n-th double point. If the double point is considered as a vanishing A-cycle of the complex curve, then the contour γ_n is a the dual compact B-cycle.

To find the subleading pre-exponential factors, one should evaluate the fluctuations around the saddle points. The variation of the action (4.20) gives

$$\delta^2 S_n(x_+, x_-) = \delta x_+ \delta x_- - \frac{1}{2}\left.\frac{dx_-}{dx_+}\right|_{-i\theta_n} (\delta x_+)^2 - \frac{1}{2}\left.\frac{dx_+}{dx_-}\right|_{i\theta_n} (\delta x_-)^2 . \tag{4.29}$$

This leads to the following prefactor in the calculation of the integral (4.19)

$$\Delta_n = \left| \left(\frac{\partial x_+}{\partial \tau}\right)_{-i\theta_n} \left(\frac{\partial x_-}{\partial \tau}\right)_{i\theta_n} \left(1 - \left(\frac{dx_-}{dx_+}\right)_{-i\theta_n} \left(\frac{dx_+}{dx_-}\right)_{i\theta_n}\right) \right|^{-1/2} , \tag{4.30}$$

where the first two factors come from the quasiclassical wave functions.

Adding together contributions to the left hand side of (4.19) of the minimal saddle contour, given in (4.22), and all isolated saddle points, we get[9]

$$e^{iS_{\rm pert}} \left[\log\Lambda + \sum_{n=1}^{\infty} \sqrt{\frac{\pi}{2}}\, \Delta_n\, e^{-S_n} \right] = e^{i\phi(E)}\, \log\Lambda\, . \tag{4.31}$$

Taking the logarithm of both sides, one gets a quasiclassical expression for the zero mode that includes the non-perturbative corrections

$$\phi(E) \approx \int_{\gamma_{\rm F}} x_-\, dx_+ - i \sum_{n>0} \frac{\sqrt{\pi}\,\Delta_n}{\sqrt{2}\,\log\Lambda}\, e^{-S_n} . \tag{4.32}$$

Finally, we can apply (2.15) to evaluate the non-perturbative terms for the free energy itself which gives

$$\mathcal{F} \approx \mathcal{F}_{\rm pert} + \frac{i\sqrt{\pi}}{2\sqrt{2}\log\Lambda} \sum_{n>0} \frac{\Delta_n}{\sin\frac{\partial_\mu S_n}{2R}}\, e^{-S_n} . \tag{4.33}$$

The classical action S_n associated with the n-th double point can be written as

$$S_n = i \int_{i\theta_n}^{-i\theta_n} x_- \partial_\tau x_+ d\tau = -2 \int^{E} \theta_n d\epsilon\, , \tag{4.34}$$

where we used the canonical transformation (2.33), and the prefactor to the exponent has the form

$$A_n \sim \frac{\Delta_n}{\sin\frac{\partial_\mu S_n}{2R}} = \left(\sin^2\frac{\theta}{R} \left[\left(\frac{\partial x_+}{\partial\tau}\right)_{-i\theta_n} \left(\frac{\partial x_-}{\partial\tau}\right)_{i\theta_n} - \left(\frac{\partial x_+}{\partial\tau}\right)_{i\theta_n} \left(\frac{\partial x_-}{\partial\tau}\right)_{-i\theta_n} \right] \right)^{-1/2} . \tag{4.35}$$

[9] Here we do not specify the contours of integration and therefore the combinatorial factors in front of the individual terms. We assume that the contribution of each saddle point enters with the factor 1/2.

The free energy is determined by (2.15) up to terms of the form $e^{-2\pi R\mu}$. The analysis of section 3 based on Toda equation shows that these corrections do not depend on the tachyon potential and therefore are given by the last term on the r.h.s. of (3.1). However there is an argument, advanced in [10], that such terms do not appear whenever a tachyon potential is switched on.

The result (4.35) gives the instanton corrections for an arbitrary tachyon deformation of the theory. Now let us restrict ourselves to the case of sine-Liouville deformation. It was already shown in [13] that in this case S_n reproduce the leading non-perturbative corrections (3.2) obtained by Toda equation. Therefore let us concentrate on the subleading contribution (4.35). Calculating the derivatives using (2.30), where only terms with $k = 1$ are present, and taking into account the defining equation (4.26) for $\theta = \theta_n$, one finds

$$A(\xi, y) = C \left\{ e^{-\frac{1}{R}\chi} \sin^2\theta \sin^2\frac{\theta}{R} \left(\left(\frac{1}{R} - 1\right) \cot\left(\frac{1-R}{R}\theta\right) - \cot\theta \right) \right\}^{-1/2}, \qquad (4.36)$$

where the overall coefficient C is found to be

$$C = \frac{i\sqrt{\pi R}}{4\sqrt{2}\log\Lambda}. \qquad (4.37)$$

This result is in complete agreement with the result (3.12) obtained by integrating Toda equation.

In particular, this confirms our claim that for any finite tachyon deformation the subleading corrections scale like $g_{\mathrm{str}}^{1/2}$ and thus they are non-analytic in the limit $\lambda \to 0$. This is due to the breakdown of the time translation invariance in any time-dependent background. As a consequence, in the first case the contributions to the saddle point approximation come from one-dimensional saddle contours, whereas in the second case they arise from isolated saddle points. Hence, the determinants of fluctuations, giving the main non-trivial contribution to the subleading correction, are one- and two-dimensional, correspondingly. This explains the difference in the power of the string coupling.

5. The instanton effects from bosonization

In the matrix model, the tachyon modes are collective excitations of fermions propagating as left and right moving waves on the Fermi surface. After second quantization, these modes form a bosonic field. In (x, p) representation of MQM, the fermions are non-relativistic and as a consequence the bosonic field is self-interacting. In contrast, as in (x_+, x_-) representation the one-particle fermionic hamiltonian becomes first order, the fermions can be exactly bosonized. This bosonization is essentially the one that occurs in Toda hierarchy, modulo some subtleties related to the fact that we are using a non-compact realization of the latter.

In this section we show that the non-perturbative corrections to the free energy can be expressed in terms of vertex operators for a boson field. The bosonic field in question describes not the whole spectrum of tachyons in the theory, but only the discrete subset that survives after the compactification. The bosonic field formalism works particularly well in the quasiclassical limit and gives a nice interpretation to the results obtained in the previous two sections. It might also help to interpret these results in terms of D-branes in $c = 1$ string theory with time-dependent background.

5.1 Second quantized fermions and density operator

We start by introducing the second quantized fermion fields

$$\begin{aligned} \hat{\Psi}_{\pm}(x_{\pm}, t) &= \int dE\, e^{\mp Et/2}\, \Psi_{\pm}^{E}\left(e^{\mp t} x_{\pm}\right) b(E)\,, \\ \hat{\Psi}_{\pm}^{\dagger}(x_{\pm}, t) &= \int dE\, e^{\mp Et/2}\, \overline{\Psi_{\pm}^{E}\left(e^{\mp t} x_{\pm}\right)}\, b^{\dagger}(E)\,, \end{aligned} \tag{5.1}$$

where the operator amplitudes $b(E)$ and $b^{\dagger}(E)$ satisfy the canonical anti-commutation relations

$$\{b^{\dagger}(E), b(E')\} = \delta(E - E')\,, \tag{5.2}$$

with all other anticommutators equal to zero. We work with only one set of operator amplitudes, since the left and right fermion operators are related by Fourier transformation and the fermion wave functions have the property (2.27). The last property followed from the fact that we absorbed the reflection phase in the definition of the wave functions, so that the fermion reflection operator acts as the identity operator on the amplitudes in E-representation.

We are interested in operators in the matrix model of the form

$$\mathcal{O}_f = \mathrm{Tr}\, f(\mathbf{X}_+, \mathbf{X}_-)\,, \tag{5.3}$$

where $f(x_+, x_-)$ is a smooth function of its two variables. The operator $\mathcal{O}_f$ is well defined as the matrices $\mathbf{X}_+$ and $\mathbf{X}_-$ commute due to the gauge field. In the second quantized formalism, this operator translates into

$$\hat{\mathcal{O}}_f = \int dx_+ dx_-\, f(x_+, x_-)\, \hat{\mathcal{U}}(x_+, x_-)\,, \tag{5.4}$$

where

$$\hat{\mathcal{U}}(x_+, x_-, t) = \tfrac{1}{\sqrt{2\pi}}\, e^{i x_+ x_-}\, \hat{\Psi}_-^{\dagger}(x_-, t)\, \hat{\Psi}_+(x_+, t) \tag{5.5}$$

is the fermion phase space density operator. The expectation value of the operator $\hat{\mathcal{U}}$, or the Wigner's function, is evaluated with respect to the thermal vacuum of the compactified theory which is defined by [3]

$$\left\langle \mu | b^{\dagger}(E)\, b(E') | \mu \right\rangle = \frac{\delta(E - E')}{1 + e^{\beta(\mu + E)}}\,. \tag{5.6}$$

In particular, for the trace of the identity operator $N \equiv \langle \mu |\, \mathrm{Tr}\, 1 | \mu \rangle$ we find, using (5.6) and (4.17),

$$N = \int dx_+ dx_-\, \langle \mu | \hat{\mathcal{U}}(x_+, x_-) | \mu \rangle = \int_{-\infty}^{\infty} \frac{dE\, \rho(E)}{1 + e^{\beta(\mu + E)}}\,. \tag{5.7}$$

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Comparing with (2.13), one reproduces the well known relation between the number of fermions and the grand canonical free energy

$$N = -\frac{\mu}{2\pi}\log\Lambda - \frac{1}{2\pi R}\partial_\mu \mathcal{F}\,, \tag{5.8}$$

where we explicitly included the non-universal cut-off dependent term. Then (2.15) allows also to write the relation of N to the reflection phase

$$\sin\left(\frac{1}{2R}\partial_\mu\right) N = -\frac{1}{4\pi R}\left(\log\Lambda + \partial_\mu\phi(-\mu)\right). \tag{5.9}$$

5.2 Bosonization formula for the fermion wave functions and the density operator

The fermionic operators can be expressed as exponents of a bosonic field with continuous spectrum of energies. It is however technically more advantageous to introduce another bosonic field with discrete spectrum associated with the possible energies in the euclidean compactified theory. If we restrict the spectrum of tachyons to be discrete as in the compactified theory, the spectrum of the bosonic fields will be also restricted to the discrete set of purely imaginary momenta $p_n = in/R,\ n\in\mathbb{Z}$. The sector of the theory spanned on these states gives a 'non-compact' realization of Toda integrable structure.[10] The bosonic field is thus spanned on the operators $\{E, \partial_E,\, t_{\pm n}, \partial_{t_{\pm n}}\}$ and has the form

$$\hat{\Phi}_\pm(x_\pm) = V_\pm(x_\pm) + \frac{1}{2}\hat{\phi} - E\log x_\pm + \hat{D}_\pm(x_\pm)\,, \tag{5.10}$$

where $V_\pm$ are the potentials (2.26) and the differential operators $\hat{\phi}$ and $\hat{D}_\pm$ are defined by

$$\hat{\phi} = -\frac{1}{R}\partial_E\,, \qquad \hat{D}_\pm(x_\pm) = \sum_{n\geq 1}\frac{1}{n}\,x_\pm^{-n/R}\,\partial_{t_{\pm n}}\,. \tag{5.11}$$

The deformed fermion wave function at level $E=-\mu$, known in the mathematical literature as Baker-Akhieser function, can be written as the expectation value of normal ordered exponential of a bosonic field. A straightforward generalization of the bosonization formula for the compact Toda hierarchy gives [52]

$$\Psi_\pm^{-\mu}(x_\pm) = (2\pi\, x_\pm)^{-1/2}\left\langle \mu\right| : e^{\mp i\hat{\Phi}_\pm(x_\pm)} : \left|\mu\right\rangle, \tag{5.12}$$

where the normal product sign : : means that all derivatives are moved to the right, and the expectation value of the differential operator $\hat{\mathcal{O}}$ is defined as $\langle\hat{\mathcal{O}}\rangle = \mathcal{Z}^{-1}\cdot\hat{\mathcal{O}}\cdot\mathcal{Z}$. From the bosonic representation of the Becher-Akhieser function we get the following operator formula for the fermion bilinears

$$\hat{\Psi}^\dagger_-(x_-)\hat{\Psi}_+(x_+) = \frac{1}{2\pi}(x_+x_-)^{-\frac{1}{2}}\left(: e^{-i\hat{\Phi}_-(x_-)} :\right)^\dagger : e^{-i\hat{\Phi}_+(x_+)} :, \tag{5.13}$$

[10] Usually in Toda hierarchy the wave functions are bi-orthogonal polynomials and the τ-functions are labeled by a non-negative integer $s\in\mathbb{Z}_+$, the degree of the polynomial [50, 51]. In our case the role of s is played by the discrete complex variable $\mu + i(n+\frac{1}{2})\frac{1}{R}$, $n\in\mathbb{Z}_+$. We use the terms compact and non-compact Toda hierarchy in the analogy with the representations of the compact $su(2)$ and the non-compact $sl(2,\mathbb{R})$.

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where the hermitian conjugation changes the normal to anti-normal ordering in the first factor. Passing to the normal ordering and subtracting a divergent term in the exponent, we finally obtain

$$\hat{\Psi}^{\dagger}_{-}(x_-)\hat{\Psi}_{+}(x_+) = \frac{1}{2\pi R}\,(x_+x_-)^{-\frac{R+1}{2R}}\, : e^{-i\hat{\Phi}(x_+,x_-)} :, \tag{5.14}$$

where we introduced the full bosonic field

$$\hat{\Phi}(x_+, x_-) = \hat{\Phi}_+(x_+) + \hat{\Phi}_-(x_-)\,. \tag{5.15}$$

Since we performed a subtraction, the overall coefficient on the r.h.s. of (5.14) was fixed by hand. This is done by comparing the quasiclassical expressions obtained below through bosonization with those that follow from the quasiclassical wave functions.

Using (5.14), one can easily write the bosonization formula for any observable of the form (5.4). In particular, for the trace of the identity operator (5.7), which gives the number of particles, one obtains

$$N = \frac{1}{(2\pi)^{3/2}R}\int \frac{dx_+ dx_-}{(x_+x_-)^{\frac{R+1}{2R}}}\, e^{ix_+x_-}\, \Big\langle \mu\Big| : e^{-i\hat{\Phi}(x_+,x_-)} : \Big|\mu\Big\rangle\,. \tag{5.16}$$

5.3 Quasiclassical limit

The quasiclassical asymptotics of the fermionic wave functions is determined, through the representation (5.12), by the quasiclassical expansion of the exponentials of the bosonic fields $\hat{\Phi}_\pm(x_\pm)$. Here we will restrict ourselves to the first two orders:

$$\Psi^{-\mu}_{\pm}(x_\pm) = \frac{1}{\sqrt{2\pi\, x_\pm}}\, e^{\mp i\Phi_\pm(x_\pm)}\, e^{-\frac{1}{2}\langle :\hat{\Phi}_\pm(x_\pm)\hat{\Phi}_\pm(x_\pm): \rangle_c}\,, \tag{5.17}$$

where

$$\Phi_\pm(x_\pm) \equiv \langle \hat{\Phi}_\pm(x_\pm)\rangle \tag{5.18}$$

is the vacuum expectation value of the field $\hat{\Phi}_\pm(x_\pm)$ and $\langle\cdots\rangle_c$ denotes the connected correlator. In the leading order the phase of the fermion wave functions are thus given by the expectation values of the left and right bosonic fields:

$$\varphi_\pm(x_\pm; -\mu) + \mu \log x_\pm \approx \Phi_\pm(x_\pm)\,. \tag{5.19}$$

In particular, the quasiclassical phase of the fermion scattering is equal to the expectation value of the zero mode $\hat{\phi}$

$$\phi = \langle \hat{\phi}\rangle\,. \tag{5.20}$$

In order to find the subleading factor in the exponential, we need the two-point correlation functions. The connected two-point correlators are generating functions for the derivatives $\partial_\mu\partial_n\mathcal{F}$ and $\partial_n\partial_m\mathcal{F}$, which were calculated in the case of sine-Liouville deformation in [35] from Toda hierarchy. The answer is actually true for any deformation and can be expressed in terms of the functions $\tau(x_\pm)$ obtained by inverting the parametric

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representation (2.30) [47].[11] Note that the functions $\tau(x_\pm)$ are given by the derivative in μ of the phase (5.19) of the fermion wave functions

$$\tau(x_\pm) = \pm\partial_\mu \Phi_\pm(x_\pm)\,. \tag{5.21}$$

The explicit expressions of the generating functions are

$$\begin{aligned}
\hat{D}_\pm(x_\pm)\,\partial_\mu \mathcal{F} &= -\frac{1}{2R}\,\partial_\mu^2 \mathcal{F} - \log\left(x_\pm\, e^{\mp\tau(x_\pm)}\right),\\
\hat{D}_\pm(x_\pm)\hat{D}_\pm(y_\pm)\,\mathcal{F} &= \frac{1}{2R^2}\,\partial_\mu^2\mathcal{F} + \log\left(x_\pm^{\frac{1}{R}} - y_\pm^{\frac{1}{R}}\right) - \log\left(e^{\pm\frac{\tau(x_\pm)}{R}} - e^{\pm\frac{\tau(y_\pm)}{R}}\right),\\
\hat{D}_+(x_+)\hat{D}_-(y_-)\,\mathcal{F} &= \log\left(1 - e^{\frac{\tau(y_-)}{R} - \frac{\tau(x_+)}{R}}\right).
\end{aligned} \tag{5.22}$$

From the first two identities (5.22) one finds the connected correlation function for the left and right chiral fields:

$$\Big\langle :\hat{\Phi}_\pm(x_\pm)\hat{\Phi}_\pm(y_\pm):\Big\rangle_c = \log\left(\pm\frac{x_\pm^{1/R} - y_\pm^{1/R}}{2(x_\pm y_\pm)^{\frac{1}{2R}}\sinh\frac{\tau(x_\pm)-\tau(y_\pm)}{2R}}\right) + \frac{1}{4R}\partial_\mu\phi\,. \tag{5.23}$$

Substituting this in (5.17), one obtains the quasiclassical form of the fermion wave functions:

$$\Psi_\pm^{-\mu}(x_\pm) = \frac{e^{\mp i\Phi_\pm(x_\pm) - \frac{1}{8R}\partial_\mu\phi}}{\sqrt{\pm 2\pi\partial_\tau x_\pm}}\,. \tag{5.24}$$

Up to the last term in the exponent, the expression (5.24) reproduces the previously obtained result (4.15) together with (4.23).

To explain the appearance of the last term, let us consider the zero modes $\phi_\pm$. To all orders, they are given by

$$e^{\mp i\phi_\pm} = \langle e^{\mp i\hat{\phi}/2}\rangle = \frac{\mathcal{Z}\left(\mu \mp \frac{i}{2R}\right)}{\mathcal{Z}(\mu)}\,, \tag{5.25}$$

so that for the total zero mode we reproduce the expression (2.15):

$$e^{i\phi} = e^{i(\phi_+ + \phi_-)} = \frac{\mathcal{Z}\left(\mu + \frac{i}{2R}\right)}{\mathcal{Z}\left(\mu - \frac{i}{2R}\right)}\,. \tag{5.26}$$

On the other hand, the quasiclassical calculation gives

$$e^{\mp i\phi_\pm} \approx e^{\mp i\langle\hat{\phi}/2\rangle - \frac{1}{8}\langle\hat{\phi}\hat{\phi}\rangle_c} = e^{\mp i\phi/2 - \frac{1}{8R}\partial_\mu\phi}\,, \tag{5.27}$$

which is, of course, agrees with the expansion of (5.25). In this way the last term appears as a correction to the zero modes of the wave functions of chiral fermions. This term reflects the freedom in the choice of the relative normalization of the left and right fermions and is cancelled in the product $\overline{\Psi_-^{-\mu}}\Psi_+^{-\mu}$.

[11] In the case of a compact Toda hierarchy, when the fermion eigenfunctions are entire functions, these relations have been obtained in [53, 54].

Now let us turn to the fermion bilinear (5.14). In the leading order, its expectation value is given by the exponential of the expectation value of the full bosonic field,

$$\langle\hat{\Phi}(x_+,x_-)\rangle = \Phi_+(x_+) + \Phi_-(x_-)\,. \tag{5.28}$$

To find the subleading contribution, one should again evaluate the two-point correlation function of $\hat{\Phi}(x_+,x_-)$. From (5.22) one finds

$$\begin{aligned}\Big\langle :\hat{\Phi}(x_+,x_-)\hat{\Phi}(y_+,y_-): \Big\rangle_c &= \log \frac{\sinh\frac{\tau(x_-)-\tau(y_+)}{2R}\ \sinh\frac{\tau(y_-)-\tau(x_+)}{2R}}{\sinh\frac{\tau(x_+)-\tau(y_+)}{2R}\ \sinh\frac{\tau(x_-)-\tau(y_-)}{2R}} + \\ &\quad + \log(x_+^{-1/R} - y_+^{-1/R}) + \log(x_-^{-1/R} - y_-^{-1/R})\,, \end{aligned}\tag{5.29}$$

which yields for the expectation value of the fermion bilinear

$$\begin{aligned}\Big\langle \mu|\hat{\Psi}^\dagger_-(x_-)\hat{\Psi}_+(x_+)|\mu\Big\rangle &= \frac{1}{2\pi R}\,(x_+x_-)^{-\frac{R+1}{2R}}\, e^{-i\Phi(x_+,x_-)-\frac{1}{2}\langle:\hat{\Phi}(x_+,x_-)\hat{\Phi}(x_+,x_-):\rangle_c} \\ &= \frac{e^{-i\Phi(x_+,x_-)}}{2\pi\sqrt{\partial_\tau x_+\partial_\tau x_-}}\ \frac{1}{2R\,\sinh\frac{\tau(x_-)-\tau(x_+)}{2R}}\,. \end{aligned}\tag{5.30}$$

The last two terms on the r.h.s. of (5.29) appear due to the normal ordering. Without normal ordering the correlation function of the gaussian field would be given only by the first term. Taking this into account, the formula (5.30) generalizes straightforwardly to the case of several fermion bilinears:

$$\langle\mu|\prod_{j=1}^n \hat{\Psi}^\dagger_-(x_-^j)\hat{\Psi}_+(x_+^j)|\mu\rangle = \prod_{k=1}^n \frac{e^{-i\Phi(x_+^k,x_-^k)}}{2\pi\sqrt{\partial_\tau x_+^k\partial_\tau x_-^k}}\ \det_{i,j}\left(\frac{1}{2R\,\sinh\frac{\tau(x_-^i)-\tau(x_+^j)}{2R}}\right). \tag{5.31}$$

We expect that this formula becomes exact if one replaces the function $\omega_\pm = e^{\tau(x_\pm)}$ by the shift operator $e^{i\partial_\mu}$ of the Toda hierarchy.[12]

Note that the second factor in the denominator of (5.30) diverges on the surface of the Fermi sea where $\tau(x_+)=\tau(x_-)$. This divergence appears due to breakdown of the quasiclassical approximation of the bosonization formulae near the Fermi sea. This divergent factor can be canceled by application of a difference operator in μ. Indeed, in the given approximation such an operator acts only on the exponent. Therefore, applying the relation (5.21), one concludes that

$$2R\sin(\frac{1}{2R}\partial_\mu)\,\Big\langle\mu|\hat{\Psi}^\dagger_-(x_-)\hat{\Psi}_+(x_+)|\mu\Big\rangle = -\frac{e^{-i\Phi(x_+,x_-)}}{2\pi\sqrt{-\partial_\tau x_+\partial_\tau x_-}}\,. \tag{5.32}$$

As a result, the r.h.s. becomes the quasiclassical expression for the product of the wave functions $\overline{\Psi_-^{-\mu}(x_-)}\Psi_+^{-\mu}(x_+)$. This allows to write a difference equation for the expectation value of any operator $\mathcal{O}_f$ of the form (5.3)

$$2R\sin(\frac{1}{2R}\partial_\mu)\,\Big\langle\mu|\hat{\mathcal{O}}|\mu\Big\rangle = -\frac{1}{(2\pi)^{3/2}}\int \frac{dx_+dx_-}{\sqrt{-\partial_\tau x_+\partial_\tau x_-}}\, f(x_+,x_-)\, e^{ix_+x_- - i\Phi(x_+,x_-)}\,. \tag{5.33}$$

[12]In the case of the two-matrix model, where the integrable structure is that of KP hierarchy, such an exact formula was derived in [21].

The non-perturbative effects to the free energy follow from this result if one specify $\mathcal{O}$ to be the identity operator ($f = 1$). Then the relation (5.9) implies

$$\frac{1}{\sqrt{2\pi}} \int dx_+ dx_- \frac{e^{ix_+ x_- - i\Phi(x_+, x_-)}}{\sqrt{-\partial_\tau x_+ \partial_\tau x_-}} = \log \Lambda \,. \tag{5.34}$$

The evaluation of this integral by the saddle point method gives the non-perturbative corrections to the zero mode ϕ and, through the relation (2.15), to the free energy itself, as it was done in section 4.3.

6. Discussion

In this paper we studied non-perturbative corrections to the partition function of the compactified $c = 1$ string theory deformed by a generic tachyon source. The flows between different backgrounds are described by a non-compact realization of Toda hierarchy. The space of all such backgrounds is parametrized by a set of Toda 'times' $\{t_{\pm k}\}$, which in the world sheet CFT have the meaning of coupling constants for the allowed marginal deformations. Each such time-dependent background is described by a complex curve, which gives the solution of Toda hierarchy in the dispersionless, or quasiclassical, limit.

We studied the dual realization of the string theory as the singlet sector of the matrix quantum mechanics, described by free fermions in upside-down oscillator potential, using chiral canonical coordinates $x_\pm \sim x \pm p$. In the fermionic system the complex curve appears as the complexified classical trajectory at the Fermi level. This curve has an infinite (for R irrational) number of double points, obtained as solutions of a transcendental equation. The leading non-perturbative corrections are associated with these double points, in the same way as in the minimal string theories. The exponents for the leading NPC are given by integral of a holomorphic differential along the compact cycles associated with these double points.

Our main new result is the expression for the subleading correction given in (4.35) for a general tachyon potential and in (4.36) for the particular case of sine-Liouville theory. We derived these results by quasiclassical analysis of the fermionic wave functions as well as by solving the linearized Toda equation in the simplest case of sine-Liouville deformation. Then we showed that these results have a natural interpretation in terms of a bosonic collective field, whose oscillator modes are in correspondence with the Toda couplings t_k. The fermion bilinears then can be represented as vertex operators of this bosonic field, eq. (5.14). The subleading corrections are proportional to the exponent of the two-point bosonic correlation function, eq. (5.30).

It is natural to expect that the NPC have their origin in the D-branes of the string theory. In order to have such an interpretation, they should be expressed in terms of world sheets with local boundary conditions. Such an interpretation exists for the eigenvalue x of the random matrix variable, which is the boundary cosmological constant for the Liouville field.

On the other hand, our results were derived in terms of the chiral phase-space coordinates, x_+ and x_-, which do not have direct interpretation in terms of local conformal boundary conditions in the world-sheet theory. Therefore, to find the meaning of the non-perturbative corrections as the effects due to D-branes, we should first translate the results obtained in the (x_+, x_-) representation back to the (x, p) representation.

Assuming that $t_k = t_{-k}$ and that there is a finite number of non-vanishing couplings, the parametric form of the curve in the (x, p) space is obtained directly from (2.30):

$$\begin{aligned} x(\tau) &= \sqrt{2}\, e^{-\frac{1}{2R}\chi} \left[\cosh\tau + \sum_{k=1}^{k_{\max}} a_k \cosh\left(\left(1-\frac{k}{R}\right)\tau\right)\right], \\ p(\tau) &= \sqrt{2}\, e^{-\frac{1}{2R}\chi} \left[\sinh\tau + \sum_{k=1}^{k_{\max}} a_k \sinh\left(\left(1-\frac{k}{R}\right)\tau\right)\right]. \end{aligned} \tag{6.1}$$

The FZZT curve is the one for the resolvent $y = w(x)$. The latter is related to the momentum p by

$$p(x) = \frac{w(e^{i\pi}x) - w(e^{-i\pi}x)}{2i}. \tag{6.2}$$

This equation is easy to solve only in the case of stationary background, where the two sides of the cut of $w(x)$ are the images of the straight lines $\mathrm{Im}\,\tau = \pm\pi$, and whose solution is given by (4.13). In the case of a time-dependent background the two sides of the cut of $w(x)$ are parametrized by the lines $\tau_\pm(t) = t \pm i\pi_1(t)$, $t > 0$, where $\pi_1(t) \to \pi$ only asymptotically at $t \to \pm\infty$. Therefore eq. (6.2) can be written more explicitly as

$$p(\tau) = \frac{w(\tau + i\pi_1(\tau)) - w(\tau - i\pi_1(\tau))}{2i}, \tag{6.3}$$

where the function $\pi_1(t)$ is obtained as the solution of the transcendental equation

$$\sin\pi_1 + \sum_{k=1}^{k_{\max}} a_k \frac{\sinh\left((1-\frac{k}{R})t\right)}{\sinh t} \sin\left((1-\frac{k}{R})\pi_1\right) = 0\,, \qquad t \in \mathbb{R}. \tag{6.4}$$

As a consequence, the parameter τ does not uniformize the curve $y = w(x)$. In general, $w(\tau)$ will have infinitely many branch points.

Considered at $t = 0$, eq. (6.4) coincides with the condition that $\partial_\tau x = 0$ for purely imaginary τ. Therefore the point $\tau = \pm i\pi_1(0)$ is a branch point for $p(x)$. It is easy to see that all purely imaginary solutions of (6.4) correspond to branch points of $p(x)$. The solutions of (6.4) can be classified by their asymptotics at infinity: $\pi_n(t) \to n\pi$.

The equation (6.4) for $\pi_n(\tau)$ is not compatible with the transcendental equation (4.25) for the parameters θ_n of the double points, except for the stationary background. Therefore, for time-dependent backgrounds the double points never occur at the branch points of the resolvent or along its cuts. This fact can be geometrically understood by considering the classical trajectory $x(i\theta)$ in imaginary time direction. The branch points are the turning points of the classical trajectory, while the double points are associated with its intersection points. Therefore, turning on a time-dependent tachyon background resolve the infinite degeneracy of the double points of the complex curve for the $c = 1$ string theory.

Now let us return to the interpretation of the non-perturbative corrections in terms of D-branes. For the leading non-perturbative effects, which are given by the closed contour integrals S_n (4.28), this was done in [13, 14] . Using the relation (D.10) of the light-cone coordinates to the spectral density and (6.2), one obtains

$$S_n = -\frac{1}{2}\oint_{\gamma_n} dx\, \big(w(e^{i\pi}x) - w(e^{-i\pi}x)\big)\,. \tag{6.5}$$

The resolvent $w(x)$ is equal to the derivative of the FZZT disk partition function with respect to the boundary cosmological constant x:

$$w(x) = \partial_x \Phi_{_{FZZT}}(x)\,. \tag{6.6}$$

Taking into account the symmetry $w(e^{i\pi}x(\tau)) = w(e^{-i\pi}x(-\tau))$ of the resolvent, this allows to identify

$$-S_n = \Phi_{_{FZZT}}(e^{-i\pi}x(i\theta_n)) - \Phi_{_{FZZT}}(e^{i\pi}x(i\theta_n))\,. \tag{6.7}$$

In [13] it was also checked in the first order in the deformation coupling λ that the leading correction is given by the ZZ disk partition function

$$-S_n = \Phi_{_{ZZ}}(n,1) \tag{6.8}$$

so that the n-th double point of the complex curve is associated to the $(n,1)$ ZZ brane. Assuming that this equation remains to be true in the full deformed theory, from (6.7) and (6.8) we conclude that the ZZ partition function is equal to the difference of two FZZT partition functions calculated from two sides of the double point. This is a generalization to the case of a general tachyon deformation of a similar identity in the non-deformed theory ($\lambda = 0$). [7, 25, 55, 27].

Note that our analysis also reveals the distinguished role of the $(1,1)$ ZZ brane. Up to now it was the only one which played a role in the interpretation of matrix models as theories of open strings. In our context it is associated with the first double point of the complex curve. One can show that this is the only double point which is situated on the physical sheet of the Riemann surface. It is natural to expect that this fact corresponds to the observation in CFT that the $(n,1)$ ZZ branes with $n>1$ possess some pathological properties [24]. However, if we are calculating the non-perturbative corrections to observables that explore lower sheets of the Riemann surface, than the other double points may become relevant.

The world-sheet interpretation of the subleading order of NPC seems to be a more complicated problem. It is expected to be related to the annulus amplitude on ZZ brane. However, the expression given for this amplitude in [16] diverges in this case. One may attempt to relate this divergence to the factor $(\log\Lambda)^{-1}$ in (4.37). But in this way we can not explain the universal non-trivial functional dependence on the couplings. On the other hand, our results are naturally expressed in terms of the two-point amplitude of a gaussian field whose left and right components are associated with the incoming and outgoing tachyons. It is not however obvious how to reformulate them in terms of the annulus open string amplitude on FZZT branes, which is defined in terms of the eigenvalue x.

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A. Relations between the density of states, the scattering phase and the free energy

Knowing the zero mode $\phi(E)$ of the fermion wave functions, one can reconstruct the density of states by introducing a completely reflecting cut-off wall at distance $\Lambda \gg \mu$ [49, 52]. The wall introduces a boundary condition at $x_+ = x_- = \sqrt{\Lambda}$

$$[\hat{S}\Psi](\sqrt{\Lambda}) = \Psi(\sqrt{\Lambda})\,. \tag{A.1}$$

Thus, one identifies the scattered state with the initial one at the wall. Applying this condition to the deformed wave functions (2.24), one obtains that it is satisfied for a discrete set of energies E_n ($n \in \mathbb{Z}$) defined by

$$\phi(E_n) - E_n \log\Lambda + V(\Lambda) + 2\pi n = 0\,, \qquad V(\Lambda) = \sum_k (t_k + t_{-k})\Lambda^{k/2R}\,. \tag{A.2}$$

From (A.2) one can find the density of the energy levels in the confined system

$$\rho(E) = \frac{\log\Lambda}{2\pi} - \frac{1}{2\pi}\frac{d\phi(E)}{dE}\,. \tag{A.3}$$

Now the free energy $\mathcal{F}(\mu, R) = \log \mathcal{Z}(\mu, R)$ of the fermionic system compactified at distance $\beta = 2\pi R$ can be calculated using (2.13) with the density (A.3). Dropping out the Λ-dependent non-universal contribution and integrating by parts, one obtains

$$\mathcal{F}(\mu) = -\frac{1}{2\pi}\int d\phi(E)\log\left(1 + e^{-\beta(\mu+E)}\right) = -R\int_{-\infty}^{\infty} dE\frac{\phi(E)}{1 + e^{\beta(\mu+E)}}\,. \tag{A.4}$$

We close the contour of integration in the upper half plane and take the integral as a sum of residues. This gives for the free energy

$$\mathcal{F}(\mu) = i\sum_{r=n+\frac{1}{2}>0}\phi\left(\frac{ir}{R} - \mu\right) \tag{A.5}$$

from which (2.15) follows.

B. Solution of (3.7)

In this appendix we solve the equation (3.7) using the ansatz (3.8). Substituting (3.8) into (3.7), keeping only the leading terms in the $\xi \to 0$ limit (which come from the terms with all or all except one derivatives acting on the exponent in (3.8)), and taking into account that $g(y)$ satisfies the equation

$$\sqrt{\alpha}\, e^{\frac{X(y)}{2R^2}}(1-y\partial_y)g(y) = \sin\left[\frac{1}{R}\partial_y g(y)\right], \tag{B.1}$$

one finds the following first order differential equation on $A(\xi,y)$:

$$\alpha\, e^{\frac{X}{R^2}}\left[2\left((1-y\partial_y)g\right)(y\partial_y+\xi\partial_\xi)\log A-(1-y\partial_y)^2 g\right] = -\frac{1}{R}\sin\frac{2g'}{R}\,\partial_y\log A-\frac{g''}{R^2}\cos\frac{2g'}{R}. \tag{B.2}$$

Taking into account the form of $A(\xi,y)$, one obtains two equations for $a(y)$ and $b(y)$

$$b'\left[2\alpha\, e^{\frac{X}{R^2}}(1-y\partial_y)g+\frac{1}{R}\sin\frac{2g'}{R}\right] = 0, \tag{B.3}$$

$$\alpha\, e^{\frac{X}{R^2}}\left[2b\left((1-y\partial_y)g\right)(1+y\partial_y\log a)-(1-y\partial_y)^2 g\right] = -\frac{b}{R}\sin\frac{2g'}{R}\,\partial_y\log a-\frac{g''}{R^2}\cos\frac{2g'}{R}.$$

The only solution of the first equation is

$$b = \text{const.} \tag{B.4}$$

Then the solution of the second equation is given by the following integral

$$\log a(y) = \int dy\,\frac{b^{-1}\left[\alpha\, e^{\frac{X}{R^2}}(1-y\partial_y)^2 g-\frac{g''}{R^2}\cos\frac{2g'}{R}\right]-2\alpha\, e^{\frac{X}{R^2}}(1-y\partial_y)g}{2\alpha\, e^{\frac{X}{R^2}}y(1-y\partial_y)g+\frac{1}{R}\sin\frac{2g'}{R}}. \tag{B.5}$$

Using (B.1), the integral can be rewritten as

$$\log a(y) = -\frac{1}{2b}\log\left|\alpha\, e^{\frac{X}{R^2}}y(1-y\partial_y)g+\frac{1}{2R}\sin\frac{2g'}{R}\right|+\int dy\,\frac{\sqrt{\alpha}e^{\frac{X}{2R^2}}\left(\frac{1}{2b}(2+\frac{y}{R^2}\partial_y X)-1\right)}{\frac{1}{R}\cos\frac{g'}{R}-2\sqrt{\alpha}\sinh\left(\frac{2R-1}{2R^2}X\right)}. \tag{B.6}$$

Then we change the integration variable to X using the equation (2.23). The second term in (B.6) becomes

$$\sqrt{\alpha}\int dX\,\frac{-\frac{2R-1}{bR}\cosh\left(\frac{2R-1}{2R^2}X\right)+\left[(\frac{1}{R}-1)e^{\frac{2R-1}{2R^2}X}-e^{-\frac{2R-1}{2R^2}X}\right]}{\frac{1}{R}\cos\frac{g'}{R}-2\sqrt{\alpha}\sinh\left(\frac{2R-1}{2R^2}X\right)}. \tag{B.7}$$

Changing the variables to $\theta = g'$ by means of (3.4), after simple algebra one arrives at the following integral

$$-\int d\theta\,\frac{\frac{1}{b}\left[\sin^2\left(\frac{1-R}{R}\theta\right)+\left(\frac{1}{R}-1\right)\sin^2\theta\right]+\frac{2R}{2R-1}\left[\left(\frac{1}{R}-1\right)^2\sin^2\theta-\sin^2\left(\frac{1-R}{R}\theta\right)\right]}{\sin\theta\sin\frac{\theta}{R}\sin\left(\frac{1-R}{R}\theta\right)} =$$

$$= \frac{1}{b}\log\left[\frac{C\,\sin\frac{\theta}{R}}{\sin\theta\sin\left(\frac{1-R}{R}\theta\right)}\right]+\frac{2R}{2R-1}\log\left[\frac{\sin\theta}{\left(\sin\frac{\theta}{R}\right)^{2-\frac{1}{R}}\left(\sin\left(\frac{1-R}{R}\theta\right)\right)^{\frac{1}{R}-1}}\right], \tag{B.8}$$

where C is an integration constant. Combining this result with the first term in (B.6) rewritten in terms of θ, one obtains

$$\log a(y) = \frac{X}{R} - 2\log\left[\frac{\sin\frac{\theta}{R}}{\sin\left(\frac{1-R}{R}\theta\right)}\right] - \frac{1}{2b}\log\left[\left(\frac{1}{R}-1\right)\cot\left(\frac{1-R}{R}\theta\right) - \cot\theta\right] - \\ -\frac{1}{b}\log\left[C^{-1}\sin\theta\sin\left(\frac{1-R}{R}\theta\right)\right]. \tag{B.9}$$

As a result, the pre-exponential factor reads

$$A(\xi, y) = C\left[\xi\,\frac{e^{\frac{X}{R}}\sin^2\left(\frac{1-R}{R}\theta\right)}{\sin^2\frac{\theta}{R}}\right]^b \times \\ \times\left\{\sin^2\theta\sin^2\left(\frac{1-R}{R}\theta\right)\left(\left(\frac{1}{R}-1\right)\cot\left(\frac{1-R}{R}\theta\right) - \cot\theta\right)\right\}^{-1/2}. \tag{B.10}$$

C. The $c = 0$ critical limit for the subleading contribution

It is well known [56] that sine-Liouville theory exhibits a critical behavior when sine-Liouville coupling becomes sufficiently large. Physically, at the critical point the fluctuations of the matter field get frozen in minima of sine potential. As a result, the system approaches the $c = 0$ CFT describing pure two-dimensional gravity.

In [10, 11] it was shown that the leading non-perturbative correction to the partition function corresponding to $n = 1$ follows the same pattern, whereas other corrections with $n > 1$ disappear in the $c = 0$ critical limit. Thus, we expect that $A_1(y)$ must exhibit a singularity as $y \to y_c$ with

$$y_c = -(2R-1)R^{-\frac{R}{2R-1}}(1-R)^{\frac{1-R}{2R-1}} \tag{C.1}$$

and, furthermore, the behavior of A_1 near this singularity should reproduce the non-perturbative effects of the $c = 0$ theory

$$A_1 \sim g_{\mathrm{str},c=0}^{1/2} \sim (y - y_c)^{-5/8}. \tag{C.2}$$

Indeed, the singularity at $y = y_c$ corresponds to a critical point of (2.23), near which the relation between y and X degenerates:

$$\frac{y_c - y}{y_c} \simeq \frac{1-R}{2R}(X - X_c)^2 + O\left((X - X_c)^3\right). \tag{C.3}$$

At the critical point $\theta_1 \to 0$ and the first two terms in the expansion of θ_1 around the singularity are

$$\theta_1(y) = \sqrt{3}(X_c - X)^{1/2} - \frac{\sqrt{3}(2R^2 - 2R + 1)}{20R^3}(X_c - X)^{3/2} + O\left((X_c - X)^{5/2}\right). \tag{C.4}$$

Due to this result, the leading term in (3.12) is

$$A_1(y) \approx C\,\frac{R^{\frac{3R-2}{2R-1}}\left((1-R)\lambda\right)^{-\frac{R}{2R-1}}}{3^{3/4}\sqrt{2R-1}}\,(X_c - X)^{-5/4}. \tag{C.5}$$

Taking into account (C.3), one finds the law (C.2) corresponding to the $c = 0$ theory.

D. Quasiclassical wave function in the x-representation

In this appendix we study the quasiclassical wave function of the deformed fermionic system in the x-representation where x is the usual fermion coordinate related to eigenvalues of MQM. The simplest way to get this function is to use the result (4.15) together with (4.23) for the same wave function in the light-cone representation. Then it is enough to apply a unitary operator relating the two representations.

This unitary operator can be represented as an integral operator with kernel defined as a solution of the following equations

$$\frac{1}{\sqrt{2}}\left(x \mp i\partial_x\right)|x\rangle\langle x_\pm| = x_\pm |x\rangle\langle x_\pm| \,, \tag{D.1}$$

$$\frac{1}{\sqrt{2}}\left(x_\pm \pm i\partial_{x_\pm}\right)\left(|x\rangle\langle x_\pm|\right)^* = x\left(|x\rangle\langle x_\pm|\right)^* . \tag{D.2}$$

It is easy to check that it is given by

$$|x\rangle\langle x_\pm| = \frac{1}{2^{1/4}\sqrt{\pi}}\, e^{\mp\frac{i}{2}\left(x^2 - 2\sqrt{2}xx_\pm + x_\pm^2\right)} . \tag{D.3}$$

Then one obtains (the same result will be for the "x_-" representation)

$$\begin{aligned}\Psi^E(x) &= \int_0^\infty dx_+\, |x\rangle\langle x_+|\Psi_+^E(x_+)\rangle \\ &= \int_0^\infty dx_+\, \frac{e^{-i\phi_+}}{2^{3/4}\pi\sqrt{\partial_\tau x_+}}\, e^{-\frac{i}{2}\left(x^2 - 2\sqrt{2}xx_+ + x_+^2 + 2\int_\infty^{x_+} X_-(x'_+)dx'_+\right)} .\end{aligned} \tag{D.4}$$

The integral can be calculated by the saddle point method. The saddle point equation reads

$$\frac{x_+ + X_-(x_+)}{\sqrt{2}} = x \,, \qquad \frac{x_+ - X_-(x_+)}{\sqrt{2}} = p \tag{D.5}$$

where $X_-(x_+)$ is the function defined by (2.30). The fluctuations of the phase around this point are $-\left(1+\frac{dx_-}{dx_+}\right)\frac{(\delta x_\pm)^2}{2}$. Altogether these results give

$$\Psi^E(x) = \frac{1}{\sqrt{2\pi|\partial_\tau x|}}\exp\left(i\int^x p(x')\,dx'\right) , \tag{D.6}$$

where the low limit of integration in the phase can be chosen to cancel the zero mode ϕ_+.

Finally, let us show how the relation, which is evident in the classical limit, between the density $\varrho(x)$ of eigenvalues and the functions $x_\pm = X_\pm(x_\mp)$ follows from the quasiclassical wave function (D.6). The density in the compactified theory can be written in terms of the fermion wave functions as follows

$$\varrho(x) = 2\int_{-\infty}^{\infty} dE\, \frac{\bar{\Psi}^E(x)\Psi^E(x)}{e^{-\beta(\mu+E)}+1} , \tag{D.7}$$

where the coefficient 2 comes from the fact that there are left and right moving fermions. Substituting the quasiclassical limit (D.6) for the wave functions and replacing the thermal factor by $\theta(-\mu - E)$, one finds

$$\partial_\mu \varrho(x) = -\frac{1}{\pi|\partial_\tau x|} . \tag{D.8}$$

From here, using the identities

$$\left.\frac{\partial x_{\mp}}{\partial E}\right|_{x_{\pm}} = \mp\frac{1}{\partial_\tau x_{\pm}}\,, \qquad \partial_\mu\big|_x = \partial_\mu\big|_{x_+} - \frac{1}{\sqrt{2}\partial_\tau x}\partial_{x_+}\big|_\mu\,, \tag{D.9}$$

one gets the well known result for the quasiclassical spectral density

$$\varrho(x) = \frac{1}{\pi\sqrt{2}}\left(x_+ - x_-\right), \qquad x = \frac{1}{\sqrt{2}}\left(x_+ + x_-\right). \tag{D.10}$$

References

[1] P. Di Francesco, P.H. Ginsparg and J. Zinn-Justin, *2-D gravity and random matrices*, *Phys. Rept.* **254** (1995) 1 [hep-th/9306153].

[2] P.H. Ginsparg and G.W. Moore, *Lectures on 2-D gravity and 2-D string theory*, hep-th/9304011.

[3] I.R. Klebanov, *String theory in two-dimensions*, hep-th/9108019.

[4] F. David, *Phases of the large-N matrix model and nonperturbative effects in 2-D gravity*, *Nucl. Phys.* **B 348** (1991) 507.

[5] S.H. Shenker, *The strength of nonperturbative effects in string theory*, presented at *The Cargese workshop on random surfaces, quantum gravity and strings*, Cargese, France, May 28 – Jun 1, 1990.

[6] J. McGreevy and H. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded*, *J. High Energy Phys.* **12** (2003) 054 [hep-th/0304224].

[7] E.J. Martinec, *The annular report on non-critical string theory*, hep-th/0305148.

[8] I.R. Klebanov, J. Maldacena and N. Seiberg, *D-brane decay in two-dimensional string theory*, *J. High Energy Phys.* **07** (2003) 045 [hep-th/0305159].

[9] J. McGreevy, J. Teschner and H. Verlinde, *Classical and quantum D-branes in 2D string theory*, *J. High Energy Phys.* **01** (2004) 039 [hep-th/0305194].

[10] S.Y. Alexandrov, V.A. Kazakov and D. Kutasov, *Non-perturbative effects in matrix models and D-branes*, *J. High Energy Phys.* **09** (2003) 057 [hep-th/0306177].

[11] S. Alexandrov, *(m,n) zz branes and the $c = 1$ matrix model*, *Phys. Lett.* **B 604** (2004) 115 [hep-th/0310135].

[12] N. Seiberg and D. Shih, *Branes, rings and matrix models in minimal (super)string theory*, *J. High Energy Phys.* **02** (2004) 021 [hep-th/0312170].

[13] S. Alexandrov, *D-branes and complex curves in $c = 1$ string theory*, *J. High Energy Phys.* **05** (2004) 025 [hep-th/0403116].

[14] V.A. Kazakov and I.K. Kostov, *Instantons in non-critical strings from the two-matrix model*, hep-th/0403152.

[15] M. Hanada *et al.*, *Loops versus matrices: the nonperturbative aspects of noncritical string*, *Prog. Theor. Phys.* **112** (2004) 131 [hep-th/0405076].

[16] D. Kutasov, K. Okuyama, J.-w. Park, N. Seiberg and D. Shih, *Annulus amplitudes and zz branes in minimal string theory*, *J. High Energy Phys.* **08** (2004) 026 [hep-th/0406030].

JHEP02(2005)023

[17] A. Sen, *Open-closed duality: lessons from matrix model*, *Mod. Phys. Lett.* **A 19** (2004) 841 [hep-th/0308068].

[18] J. de Boer, A. Sinkovics, E. Verlinde and J.-T. Yee, *String interactions in $c = 1$ matrix model*, *J. High Energy Phys.* **03** (2004) 023 [hep-th/0312135].

[19] D. Gaiotto and L. Rastelli, *A paradigm of open/closed duality: Liouville D-branes and the Kontsevich model*, hep-th/0312196.

[20] E.J. Martinec, *Matrix models and 2D string theory*, hep-th/0410136.

[21] J. Maldacena, G.W. Moore, N. Seiberg and D. Shih, *Exact vs. semiclassical target space of the minimal string*, *J. High Energy Phys.* **10** (2004) 020 [hep-th/0408039].

[22] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary liouville field theory, I. Boundary state and boundary two-point function*, hep-th/0001012.

[23] J. Teschner, *Remarks on liouville theory with boundary*, hep-th/0009138.

[24] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.

[25] K. Hosomichi, *Bulk-boundary propagator in Liouville theory on a disc*, *J. High Energy Phys.* **11** (2001) 044 [hep-th/0108093].

[26] B. Ponsot and J. Teschner, *Boundary Liouville field theory: boundary three point function*, *Nucl. Phys.* **B 622** (2002) 309 [hep-th/0110244].

[27] B. Ponsot, *Liouville theory on the pseudosphere: bulk-boundary structure constant*, *Phys. Lett.* **B 588** (2004) 105 [hep-th/0309211].

[28] T. Takayanagi and N. Toumbas, *A matrix model dual of type-0B string theory in two dimensions*, *J. High Energy Phys.* **07** (2003) 064 [hep-th/0307083].

[29] M.R. Douglas *et al.*, *A new hat for the $c = 1$ matrix model*, hep-th/0307195.

[30] E. Martinec and K. Okuyama, *Scattered results in 2D string theory*, *J. High Energy Phys.* **10** (2004) 065 [hep-th/0407136].

[31] J. Polchinski, *Combinatorics of boundaries in string theory*, *Phys. Rev.* **D 50** (1994) 6041 [hep-th/9407031].

[32] B. Eynard and J. Zinn-Justin, *Large order behavior of 2-D gravity coupled to $d < 1$ matter*, *Phys. Lett.* **B 302** (1993) 396 [hep-th/9301004].

[33] G.W. Moore, *Gravitational phase transitions and the sine-gordon model*, hep-th/9203061.

[34] V. Kazakov, I.K. Kostov and D. Kutasov, *A matrix model for the two-dimensional black hole*, *Nucl. Phys.* **B 622** (2002) 141 [hep-th/0101011].

[35] S. Alexandrov and V. Kazakov, *Correlators in 2D string theory with vortex condensation*, *Nucl. Phys.* **B 610** (2001) 77 [hep-th/0104094].

[36] S.Y. Alexandrov, V.A. Kazakov and I.K. Kostov, *Time-dependent backgrounds of 2D string theory*, *Nucl. Phys.* **B 640** (2002) 119 [hep-th/0205079].

[37] I.K. Kostov, *Conformal field theory techniques in random matrix models*, hep-th/9907060.

[38] R. Dijkgraaf, A. Sinkovics and M. Temurhan, *Universal correlators from geometry*, *J. High Energy Phys.* **11** (2004) 012 [hep-th/0406247].

JHEP02(2005)023

[39] A. Boyarsky, B. Kulik and O. Ruchayskiy, *Classical and quantum branes in $c = 1$ string theory and quantum hall effect*, hep-th/0312242.

[40] A. Boyarsky, V.V. Cheianov and O. Ruchayskiy, *Fermions in the harmonic potential and string theory*, *J. High Energy Phys.* **01** (2005) 010 [hep-th/0409129].

[41] G.W. Moore, N. Seiberg and M. Staudacher, *From loops to states in 2-d quantum gravity*, *Nucl. Phys.* **B 362** (1991) 665.

[42] A. Jevicki, *Development in 2-d string theory*, hep-th/9309115.

[43] G.W. Moore, *Double scaled field theory at $c = 1$*, *Nucl. Phys.* **B 368** (1992) 557.

[44] E.W. Barnes, *The theory of the double gamma function*, *Philos. Trans. Roy. Soc.* **A196** (1901) 265.

[45] J. Polchinski, *What is string theory?*, hep-th/9411028.

[46] R. Dijkgraaf, G.W. Moore and R. Plesser, *The partition function of 2d string theory*, *Nucl. Phys.* **B 394** (1993) 356 [hep-th/9208031].

[47] I.K. Kostov, *String equation for string theory on a circle*, *Nucl. Phys.* **B 624** (2002) 146 [hep-th/0107247].

[48] C.G. Callan Jr. and S.R. Coleman, *The fate of the false vacuum, 2. First quantum corrections*, *Phys. Rev.* **D 16** (1977) 1762.

[49] S.Y. Alexandrov, V.A. Kazakov and I.K. Kostov, *2d string theory as normal matrix model*, *Nucl. Phys.* **B 667** (2003) 90 [hep-th/0302106].

[50] K. Ueno and K. Takasaki, *Toda lattice hierarchy: in 'Group representations and systems of differential equations' Adv. Stud. Pure Math.* **4** (1984) 1.

[51] K. Takasaki, *Adv. Stud. Pure Math.* **4** (1984) 139.

[52] I.K. Kostov, *Integrable flows in $c = 1$ string theory*, *J. Phys.* **A 36** (2003) 3153 [hep-th/0208034].

[53] P. Wiegmann and A. Zabrodin, *Large-N expansion for normal and complex matrix ensembles*, hep-th/0309253.

[54] R. Teodorescu, E. Bettelheim, O. Agam, A. Zabrodin and P. Wiegmann, *Semiclassical evolution of the spectral curve in the normal random matrix ensemble as Whitham hierarchy*, *Nucl. Phys.* **B 700** (2004) 521 [hep-th/0407017].

[55] J. Teschner, *On boundary perturbations in liouville theory and brane dynamics in noncritical string theories*, *J. High Energy Phys.* **04** (2004) 023 [hep-th/0308140].

[56] E. Hsu and D. Kutasov, *The gravitational Sine-Gordon model*, *Nucl. Phys.* **B 396** (1993) 693 [hep-th/9212023].